MPDA
Master's degree
Parametric Design in Architecture

UNIVERSITAT POLITÈCNICA DE CATALUNYA
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Rationalisation of a Torus Surface Through Planar Paneling Methods

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Abstract

This paper presents a brief survey of the planar remeshings in the context of rationalisation of a torus surface. The study describes the evaluation methods of some necessary fabrication constraints and compares the three main fabrication methods in the context of those metrics. In the study, most of the mesh geometries are generated by UV Mapping technique, and hence, includes information over the basics of this approach. Throughout the sections, the implementation of each and every step is also explained in detail. The purpose of this research is to contribute a guide study for planar mesh generation, its analysis and the comparative presentation for the essential metrics of the rationalisation process.

Keywords: Planar Remeshings, UV Mapping, Rationalisation, Planarization, Metric Evaluation, Deviation Statistics, Torus

1. Introduction

Meshes have become pervasive form of representing a geometric object in the field of computational modelling. The most frequent used meshes are polygonal and polyhedral meshes. In either case, the mesh approximates a geometric domain through an arrangement of 'vertices', 'edges' and 'faces' that combine to define the shape of the desired objects. A 'vertex' is a point that describes the corners or intersections of a geometrical face and may also be assigned additional features such as colour and texture coordinates. An 'edge' is a connection between two vertices, while a 'face' is a closed set of edges describing a geometrical shape. [1]

Meshes can be subject to several classifications due to the criterion chosen. Pertaining to the topological shape of the faces, the common bidimensional meshes are quadrilateral and triangular meshes. Additionally, some different element-based meshes consisting of hexagonal or various irregular polygonal faces are also developed. In case of all elements being uniform or having precisely the same aspect ratio, then a mesh is called to be isotropic. Antithetically, the geometry is anisotropic when different element shapes or significant variance in aspect ratio is observed amongst the faces of the mesh.

In terms of the connectivity, meshes are divided into two main categories, namely structured and unstructured meshes. Structured meshes have regular connectivity, meaning the neighborhood relations occur upon a repetitive pattern. On the other hand, unstructured meshes are characterized by irregular connectivity, where the structure pattern is not expressed as a two or three dimensional array. [2]

1.1. A Brief Introduction to Planar Remeshings

This study mainly focuses on isotropic structured meshes of quadrilateral, triangular and hexagonal faces and examines the pros and cons of these three basic remeshings on a torus surface. Moreover, comparingly a distinct kind of remeshing i.e. planar Ngons, is also included, much as a practice and not necessarily involved in the comparisons held in the further steps of the study.

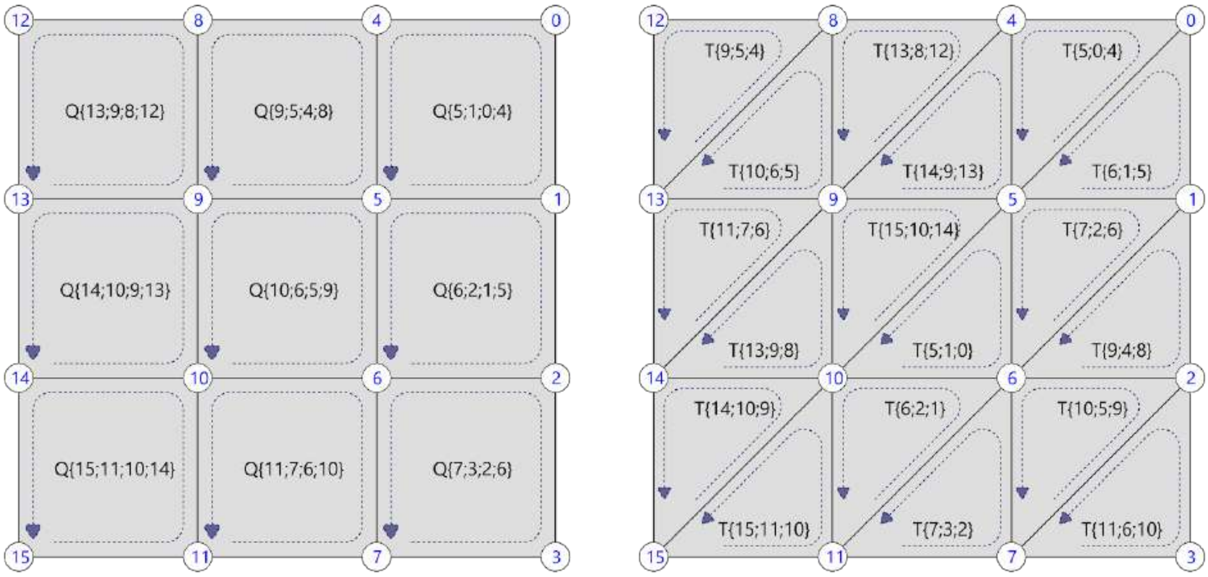


Figure 1: LHS image represents an isotropic quadrilateral mesh as well as its connectivity. RHS image shows an isotropic triangular mesh and its connectivity.

Figure 1 represents a quadrilateral mesh and a triangular mesh respectively. The quadrilateral mesh consists of 9 squares to tile a 3x3 grid where triangular mesh encompasses the same plane with 18 triangular panels in total. White circles represent the vertices of the meshes where the black coloured lines show the edges. Gray filled elements are the faces. The arrows visualize the turn of the mesh faces. In other words, following the arrows we visualise the connectivity of the mesh. The letter Q represents the faces being quadrilaterals while the letter T denotes the faces being triangles. In terms of the vertex degree, a structured isotropic quadrilateral mesh has a valence of 4 while a structured triangular mesh has a valence of 6. Further down, these two remeshings will be mentioned as Quad and Tris respectively.

Hexagonal remeshings tile a plane with regular polygons by making sure all of the edge angles around a certain vertex total 360 degrees. Pottman *et. al.* [3]. Based on this fact, Figure 2 represents a regular hexagonal mesh and its connectivity where each hexagonal face consists of 6 triangular sub-faces heading to the centre point of the hexagons. This notable difference in the structure of hexagonal mesh occurs pertaining to the development of the computational tools.[4] The hexagonal remeshings explored in this study has the same connectivity pattern represented in Figure 2 yet, of course, each of the hexagonal tile has taken into consideration as one unit panel. Further down, this remeshing will be mentioned as Hex in short. Comparingly, a regular Hex remeshing has the minimum vertex degree that is 3.

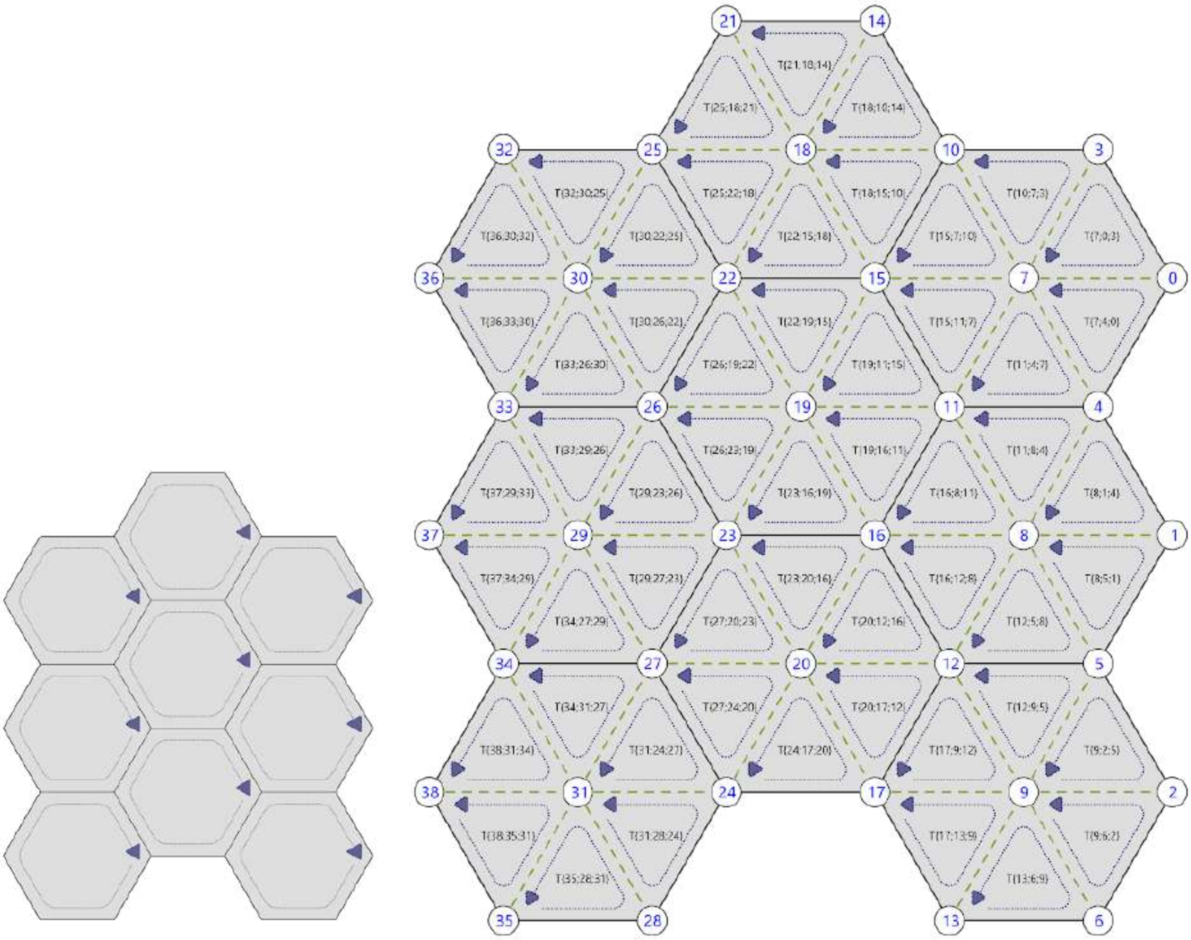


Figure 2: LHS figure represents the structure of a hexagonal grid. RHS figure shows a regular hexagonal mesh as well as its connectivity. The difference in the structure of polygonal faces having more than 4 edges occurs on account of the fact that the 3D modelling software mainly generate quadrilateral and triangular faces basically to simplify the rendering process, however, more complex polygonal elements are also possible if necessary.[4]

Planar Ngons is classified as anisotropic remeshing, may either have regular or irregular connectivity (see Fig. 3) and is only comparable with the meshes mentioned above in the context of planarity.

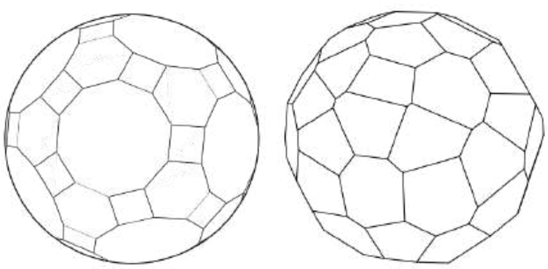


Figure 3: Ngon meshes, regular (LHS) and irregular (RHS) polygon mesh [2]

Instead of a two or three dimensional array, this kind of meshes are generated by clustering mesh faces pertaining to the normal orientation and/ or the position of the faces. The representation of the given surface with this kind of remeshing may vary due to the superiority given to either of these two parameters during the generation process. In Figure 4, three different clustering are shown where they all represent the same torus surface.

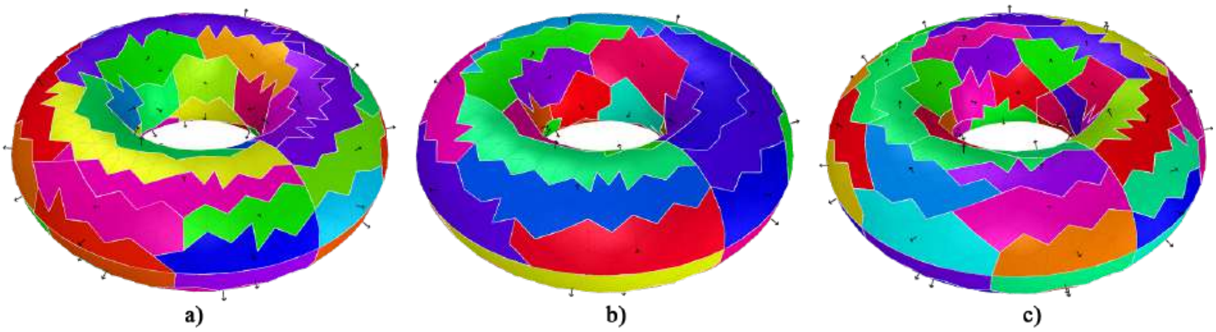


Figure 4: Various mesh face clusters applied on the same torus surface. a) 60 clusters out of 1454 triangular faces. In this clustering the two parameters, the face normals and the position of the faces are equally important. b) 42 clusters out of 1454 triangular faces. In this version the position of the faces are more dominant than the orientation of the face normals. c) 72 clusters out of 1454 triangular faces. In this version In this version, the orientation of the face normals are more dominant than the position of the faces.

1.2. A Brief Introduction to Metrics

This chapter explains the metrics taken into consideration throughout the study and gives compact information of the evaluation methods for each metric. As the main purpose of the study is to compare the isotropic form of the meshes mentioned, all of the remeshings are generated on a torus surface. To be more precise, a torus is a special rotational surface and its euler characteristic is zero, meaning no singularities are necessary while representing it as a mesh. Moreover, a torus itself consists of all kinds of surface curvature (see Fig.5) and hence, makes it possible to observe the panel variation occurring upon the curvature restrictions.

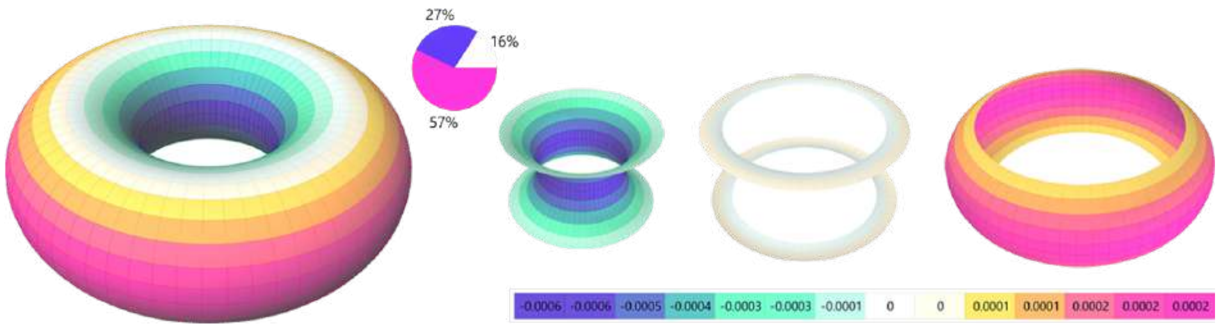


Figure 5: The total area of the surface *T* is 12,77 m2. The total area of anticlastic curvature is 3,42 m² while the synclastic curvature area is 7,34 m². The area of the flat region is calculated as 2,01 m² in total. The pie chart shows the percentage of the different curvature areas.

The torus is generated over the parametric equation seen below:

$$x = (s + r * cos(v)) * cos(u) \tag{1}$$
$$y = (s + r * cos(v)) * sin(u)) \tag{2}$$
$$z = (r * sin(v)) \tag{3}$$

for $u, v \in [0, 2\pi)$ .Pottman *et. al.* [3] where s denotes $r_1 + r_2$ and r represents r_2 . (see Fig.6) Since $s > r$, the resulting shape is a ring torus.The total surface area is 12,77 m². Further down, the torus surface is mentioned as *T* in short.

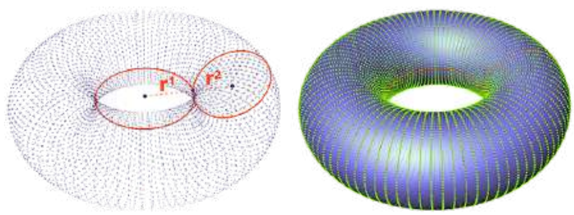


Figure 6: Ring Torus (s > r) $r_1 \approx 255$cm, $r_2 \approx 252$ cm $T_{area} \approx 12{,}77$m²;u=64, v=24 Special Rot. Surface

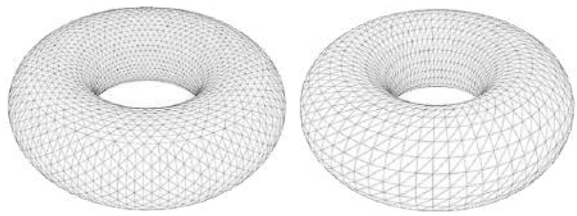


Figure 7: Depending on the generation method, the tiling pattern may vary as seen in the images above. Both of the geometries are triangular meshes representing the surface T, yet the mesh seen on the left side has vertex degree differences when compared to the mesh on right.

The quad remeshing is generated by setting the vertices array over the UV points obtained out of the equation. For the tris remeshing, the same logic is followed. The hexagonal remeshing is computed by applying dual graph over Tris and last, the Ngons remeshing is generated with K-Means Clustering.[5]. The methods used to generate the meshes are intentionally emphasized as different approaches may result in alike panelization and consequently clash with the purpose and also the observations discussed in this paper. (see Fig.7)

The metrics examined in the context of rationalising the surface *T*, are surface fairness, planarity and area deviation of the panels. Furthermore, edge length variation and aspect ratio distortion will follow those as additional evaluation criteria. Hereby, it is necessary to underline that the results obtained out of the comparisons may only be accurate within the bounds of tori surfaces. Also, during the study, the structural properties of the meshes are neglected and hence, the comparisons are only concerned over the panelization matters.

1.2.1. Surface Curvature Evaluation

For this evaluation, the surface *T* is divided into a certain amount of sub-domains and each of the domain is coloured due to the Gaussian curvature calculated at the center point of the domains. (see Fig. 5) Basically, the warm colours represent the positive curved region ($Kg > 0$) where the blueish colours show the negative curved parts ($Kg < 0$). The whitish colours represent the "almost" planar regions of *T*.

1.2.2. Surface Curvature Evaluation on Meshes

The representation of the surface curvature on mesh faces is done by first finding the surface closest point corresponding to the center point of the mesh face and then colouring the face due to the Kg value obtained at the point found on the surface. (see Fig.8)

1.2.3. Mesh Normals

This representation shows the normal orientation of the mesh faces. (see Fig.9) Apart from being a metric, mesh normal colouring is mainly used to visualize whether all of the face normals point to a consistent direction or not. If the colours occur to be harmonious, that means the mesh normals are unified. This feature is particularly important for rapid prototype printing.

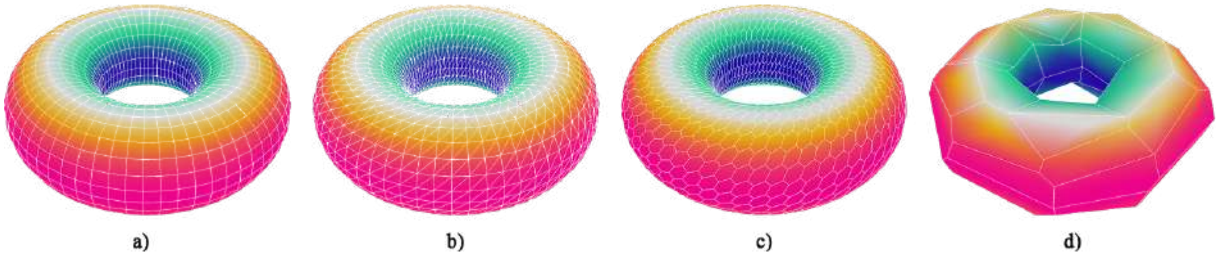


Figure 8: The figures show the surface curvature on mesh faces for each remeshing.
a) Quad remeshing b) Tris remeshing c) Hex remeshing d) Ngons remeshing

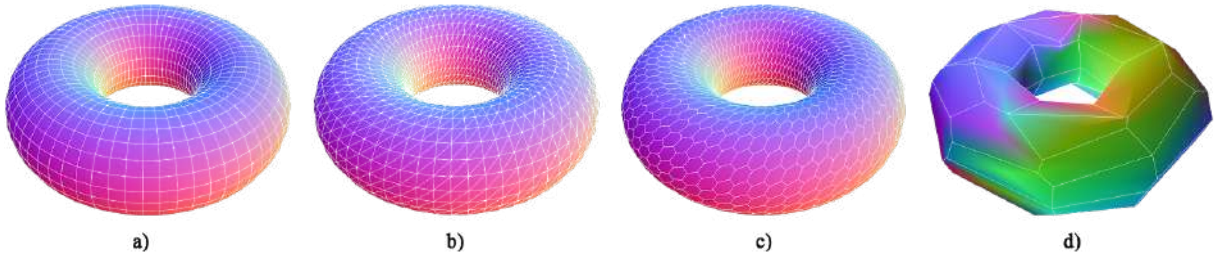


Figure 9: The meshes seen above are coloured due to the face normals.
a) Quad remeshing b) Tris remeshing c) Hex remeshing d) Ngons remeshing

1.2.4. Surface Fairness

Surface fairness is a quality metric for aesthetic assessment of the rationalisation. As there are many criteria and technologies to evaluate the fairness, in this study it is simply calculated by finding the total distance between the mesh vertices and the corresponding surface points. (see Fig.10)

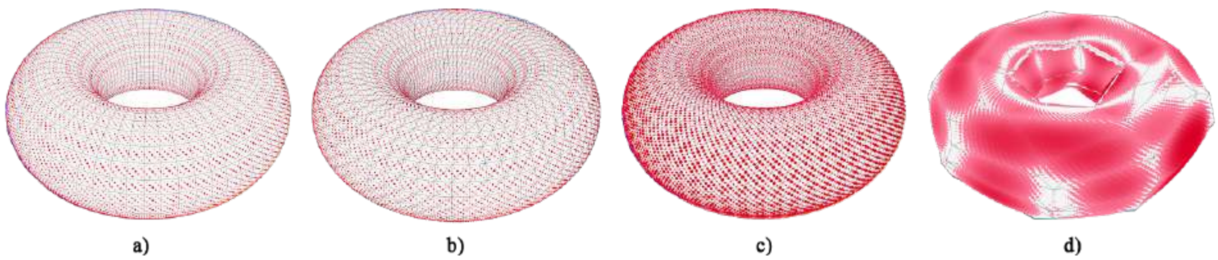


Figure 10: The images above show the surface fairness deviation. The mesh vertices are represented as spheres which their radius vary in respect to the distance between the vertex and the closest point found on surface *T*.
a) Quad remeshing b) Tris remeshing c) Hex remeshing d) Ngons remeshing

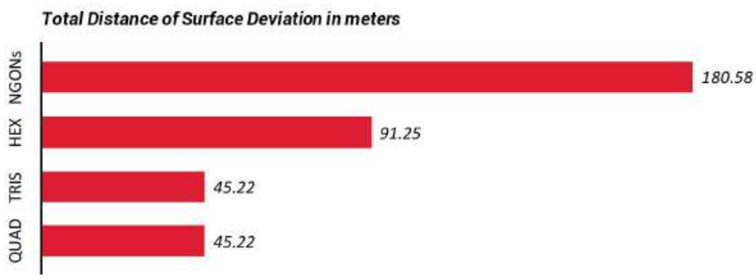


Figure 11: The chart shows the total distance of surface deviation for every remeshing.

In this particular case, Hex remeshing has the maximum amount of vertices. Therefore, this total of points is taken as the measure (18.432 vertices in number) to compare the surface fairness for all of the remeshings. Figure 11 shows the comparison chart of the total distances.

1.2.5. Planarity

Material limitations and fabrication costs generally clash with applying surface fairness at its best. To this respect, planarization is one of the ultimate steps of rationalisation process. The evaluation of planarity slightly differs due to the shape of the panel. In terms of quadrilateral panels, the first three vertices of each mesh face is introduced to output the fitting plane and next, the distance between the

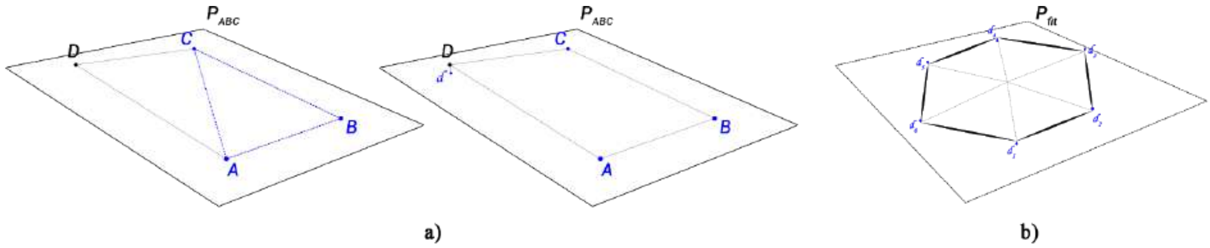


Figure 12: Three points determine a plane. Therefore, triangular faces are inherently flat.
Quadrilateral faces may either be planar or skew and faces with more vertices follow similarly.
a) *d* denotes the distance which is taken as the measure of planarity for quadrilateral panels.
b) The total of distances is taken as the measure of planarity for the rest of polygons.

fourth vertex and its projection on the plane fit is measured. Thus and so, the planarity of each face is calculated. In terms of other polygonal faces which have more vertices than four, the evaluation logic is quite similar. In this case, the best fitting plane to all of the face vertices is computed and the total distance between the vertices and their projection on the fitting plane is taken as the measure for the planarity of the face. (see Fig.12) The flatness formula of a polygon is shown below:

$$\square_{nGon} = 1/_{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} d_i \qquad (4)$$

The planarity evaluation of the four remeshings are shown in Figure 13.

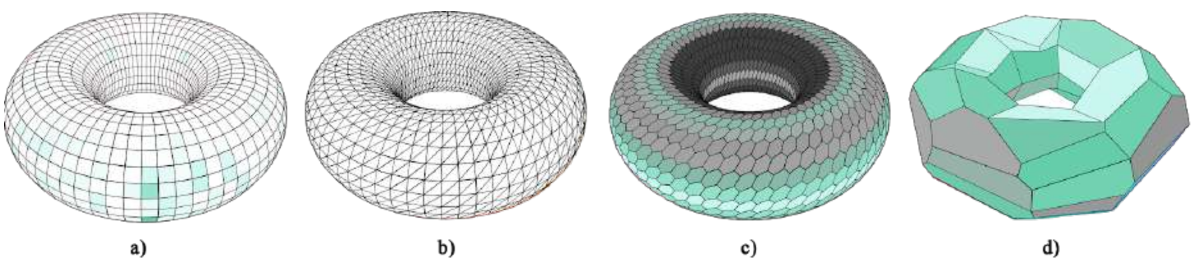


Figure 13: The images above represent the planarity of each mesh.
As colours differ from white and darken, to that extent the panel is non-planar.
a) Quad remeshing b) Tris remeshing c) Hex remeshing d) Ngons remeshing.

1.2.6. Area Deviation

Material waste is a major measure to evaluate the success of construction-aware design processes. Thereby, the area deviation is a helpful metric to discuss the necessity of optimizing the mesh as more simple and repetitive elements. It is basically represented as mesh faces being coloured referring to their area values. (see Fig.14)

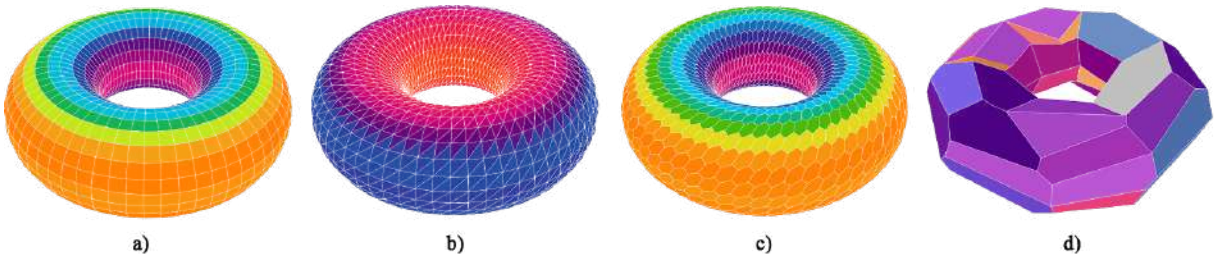


Figure 14: The images above represent the variance of the panel areas.
a) Quad remeshing b) Tris remeshing c) Hexl remeshing d) Ngons remeshing.

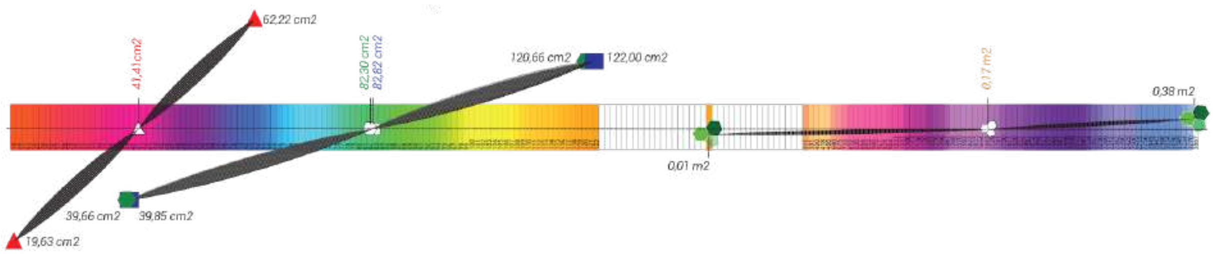


Figure 15: Legend for area evaluation

The area distribution of the remeshings is evaluated over a global domain which covers each and every area value of the four remeshings. Figure 15 shows the aforesaid domain in which the range of panel areas of each remeshing are crossed with a line (in ascending behaviour) to visualise the local domains. Also, the average area for every remeshing can be followed by referring to the values marked on the middle axis. The height of the crossing lines varies due to the total panel count of the remeshing. (Notice that Tris remeshing range is represented at two-fold height compared to Quad and Hex remeshings while Ngon remeshing covers significantly lower height.)

2. First Observations over the Metrics

Excluding the Ngon remeshing, meshes of the previous section are generated over the same amount of UV Points (1.536 in number) of the surface T and none of the any other properties match for one and another. In terms of total count of the panels, Hex remeshing has the same amount of panels with Quad, where Tris remeshing has the double count in total. Due to its generation method, Ngons remeshing is not comparable to the rest.

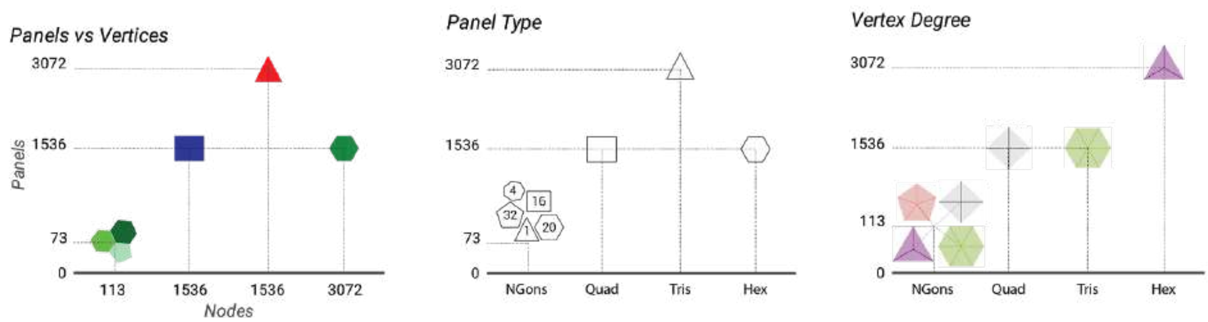


Figure 16: Comparison charts for panel-node count,panel type and vertex degree

Pertaining to the amount of the nodes, Hex remeshing requires double count of nodes for the same total of panels compared to Quad remeshing. Yet in Hex remeshing, the vertex degree for all nodes are 3 where all of the nodes in quad has a valence of 4. Tris remeshing requires the same total of nodes with Quad, however, the vertex degree for Tris is 6. (see Fig.16-panel vs. nodes and vertex degree charts) Inherently, the average area of panels in each remeshing have the same mathematical relation with the total counts of the panels.

In terms of planarity, Quad remeshing results as slightly 100% planar representation of surface *T*, yet the course version of Hex and Ngons remeshings ultimately will require planarization of the geometries when the fabrication constraints are considered. Tris remeshing is inherently %100 planar.

In terms of the area distribution, the most homogeneous result is seen on Tris remeshing, where Quad and Hex remeshings follow it as having precisely equal range of areas. Comparingly, Ngons remeshing has an extensive domain in terms of area variance.

Owing to their generation method, the first three remeshings consist panels of same topological shape. In this particular case, Ngon remeshing has both triangular, quadrilateral, pentagonal and hexagonal faces. (see Fig.16- panel type chart) Besides the panel type , also the vertex neighbours for Ngons remeshing is represented. (see Fig.17) For valence evaluation, all vertex neighbour edges are grouped for each identical vertex of the mesh and due to the total count of adjacent edges, each vertex is highlighted with colours seen on vertex degree legend shown in Figure 17. Since the meshes are generated on a torus, the aspect ratio and area deviation varies on poloidal direction in linear behaviour.

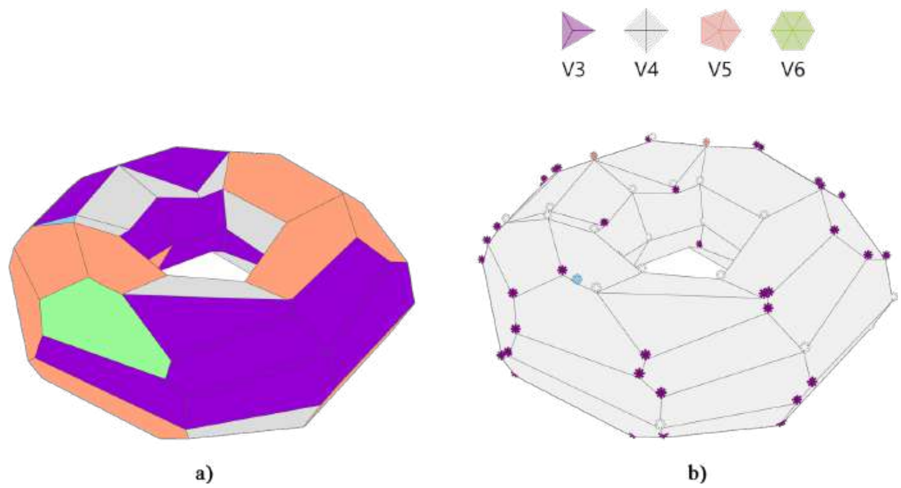


Figure 17: a) The image represents the polygon variation of the Ngon mesh.
b) The image shows the vertex degree. The legend above shows all of the vertex degrees observed on all of the four remeshings.

Respecting the first observations, each remeshing has advantages and disadvantages over several arguments. For example, if the main concern is planarity, in other words flatness , then Tris remeshing will surely be %100 superior to the rest of the meshes. But if the production budget is more concerned on significantly less complex connections, even though its planarization is more grasping, the Hex remeshing will come to the fore as the design decision. However, none of those statements is accurate in terms of comparing those geometries to one another since no metric input has been a constraint during the generation of the meshes.

Further down, a methodology is suggested for equalizing the area deviation and the total count of panels for the first three remeshings(Quad, Tris and Hex). To this respect, the rest of the metrics are evaluated and compared with the same recipes and representations explained in Section 1.2.

3. Metric Comparison of Planar Remeshings

3.1. The Methodology

In order to equalize the area deviation, first, each identical unit of those remeshings (as planar) is compared in regards to their area. The aim is to examine the numeric relation between the area property of all. The three unit surfaces are decomposed by means of a unit equilateral triangle. Due to the representation seen in Figure 18, the areas of each unit shape can be considered as 8a, 4a, and 6a respectively where a is equal to the area of the unit triangle.

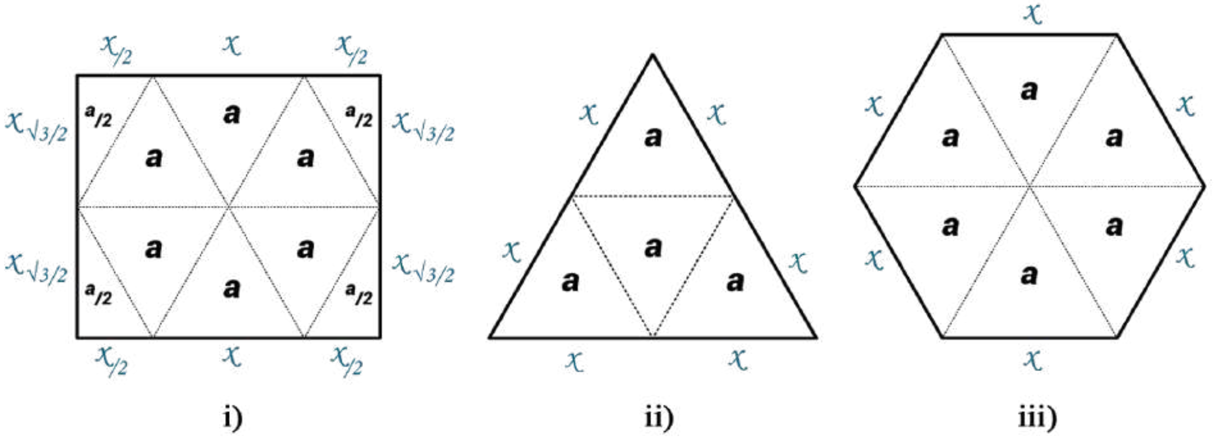


Figure 18: Decomposition of the unit faces by means of a unit equilateral triangle.
i) Unit Quad ii) Unit Tris iii) Unit Hex

If the shapes above is thought to have the same area, then it is possible to find the edge lengths for each shape by referring to the area formula of equilateral triangle that is:

$$a = \sqrt{3}/4x^2 \tag{5}$$

where a denotes the area and x represents the edge length of the equilateral triangle. Due to the factor values found (8, 4, 6), the area for every unit shape is assumed to be $48\sqrt{3}$ cm². Here it may be important to note that no equilateral triangle with its vertices having integer coordinates can be drawn in cartesian-plane. Therefore, the area of an equilateral triangle on cartesian plane cannot be an integer value.

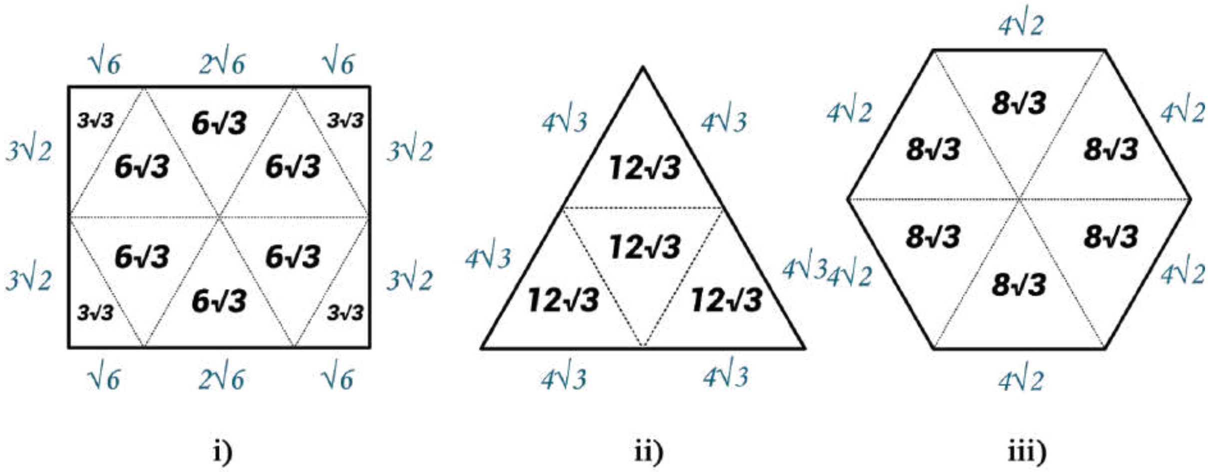
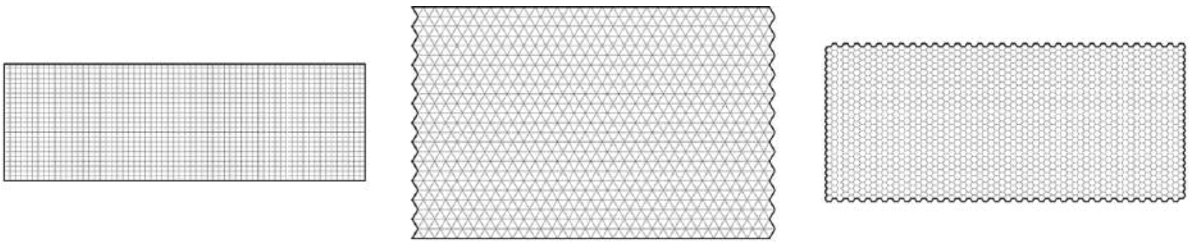


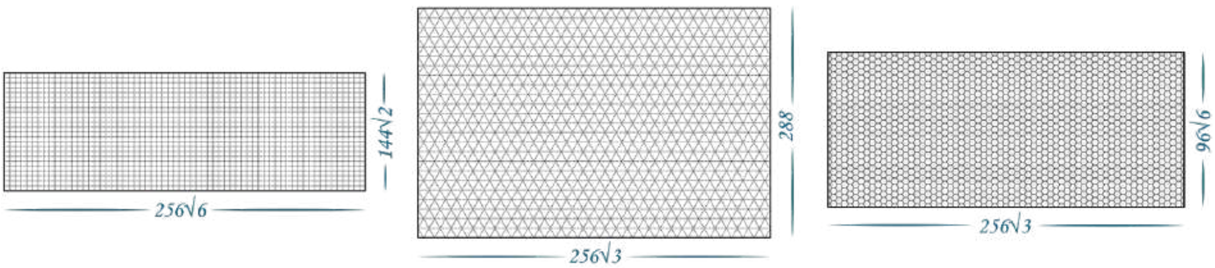
Figure 19: Edge length variation of the unit panels which have the same surface area.
i) Unit Quad ii) Unit Tris iii) Unit Hex

After finding the edge for each unit panel (see Fig.19), the grids having the same amount of cells are generated as to be mapped on the torus surface. (see Fig.20)



i) **ii)** **iii)**

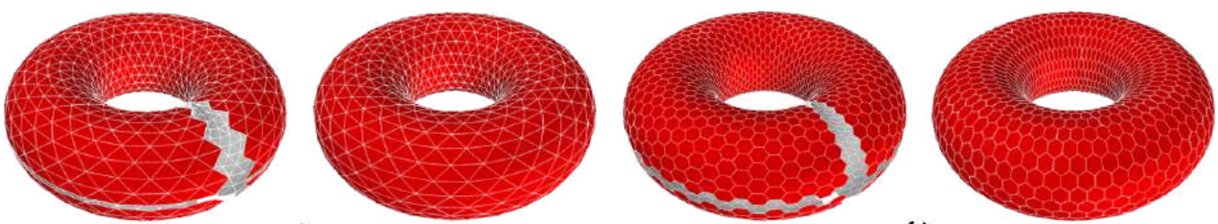
Figure 20: i) Planar Quad Mesh ii) Planar Tris Mesh iii) Planar Hex Mesh



i) **ii)** **iii)**

Figure 21: The planar meshes after the boundary refinement of Tris and Hex
i) Planar Quad Mesh ii) Planar Tris Mesh iii) Planar Hex Mesh

Before mapping the information of 2D-Domains, the boundary of Tris and Hex 2D-Domains is trimmed as regular rectangles by avoiding any change in the total area of the 2D-domains.(see Fig.21) The need for the trimming step is explained in Figure 22. Without refinement, the resulting meshes had missing faces around the seams of the torus. While trimming, the data structure of the grids are preserved to make use of the regular pattern after mapping, especially in case of mesh refinement.



a) **b)**

Figure 22: The resulting meshes on the torus with and without refining 2D-Domain boundary
a)Tris remeshing b) Hex remeshing

3.2. UV Mapping (From 2D to 3D)

The basic idea of UV Mapping is to map a 3D surface to a 2D Domain or vice versa. It is simply to have an interdimensional mapping and very often used for texture mapping. More recently, it's been used for fabrication and even for running image processing algorithms on the 2D Domain. UV Mapping has become a core tool in geometry processing as working on 2D is way easier than working on 3D. Figure 23 shows the orientation of both 2D and 3D domains. The longer edge of the 2D Domain is mapped on the toroidal direction of the torus. Thus, the information on the shorter edge generates the poloidal direction of the 3D- Domain. Lastly, Figure 24 represents where each quarter of the 2D-Domain is mapped on the 3D-Domain. Through the mapping map (see Fig.24a), it is also possible to follow the different curvatures of the torus on 2D-Domain.

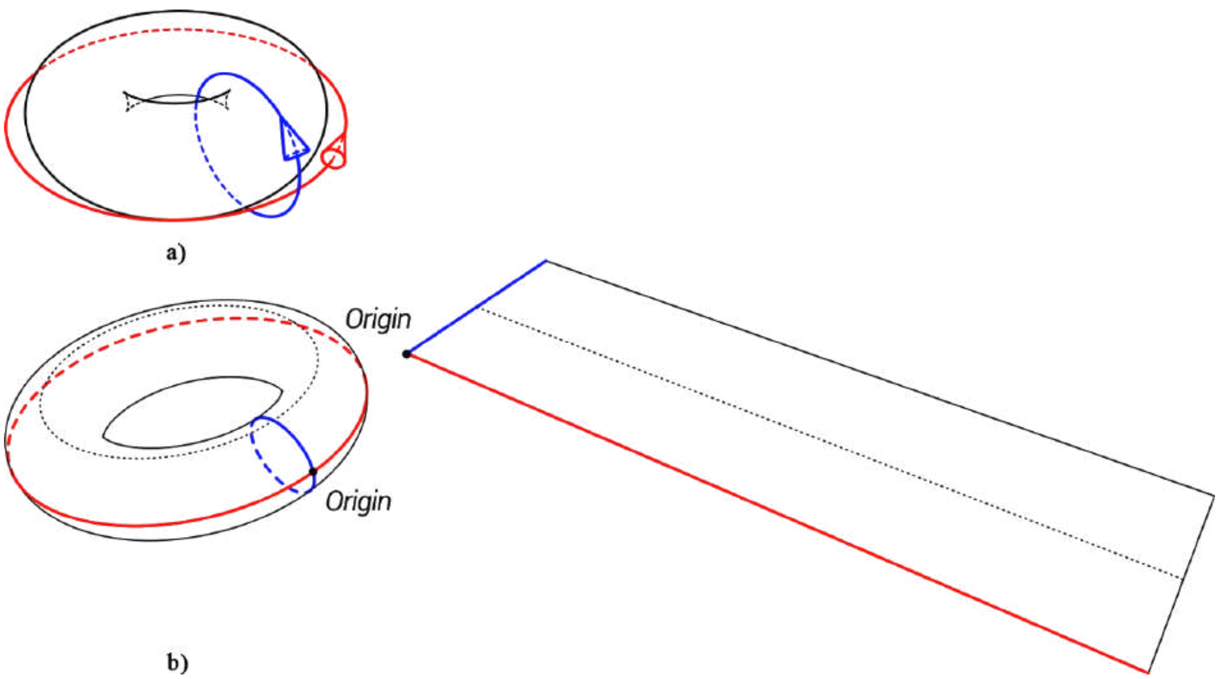


Figure 23: The orientation of 2D information on 3D domain. a) Toroidal direction (red arrow) and Poloidal direction (blue arrow) b) The image represents the UV Map principle used in this study.

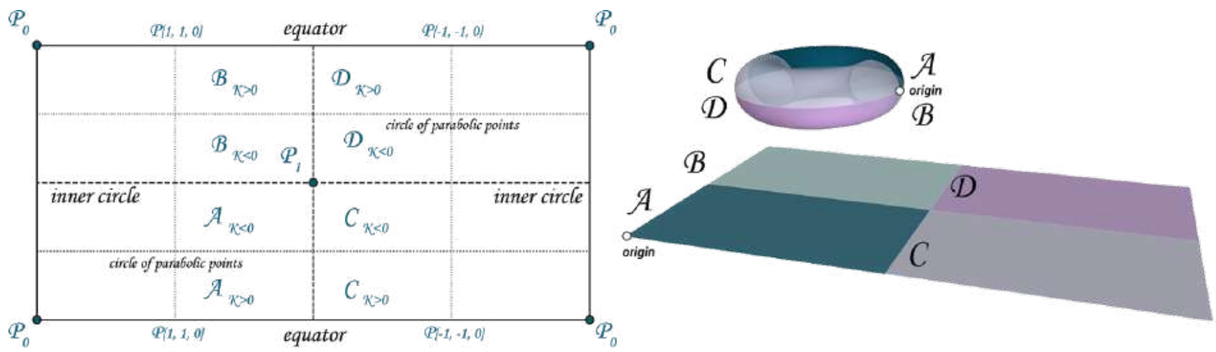


Figure 24: The orientation of 3D information on 2D domain.

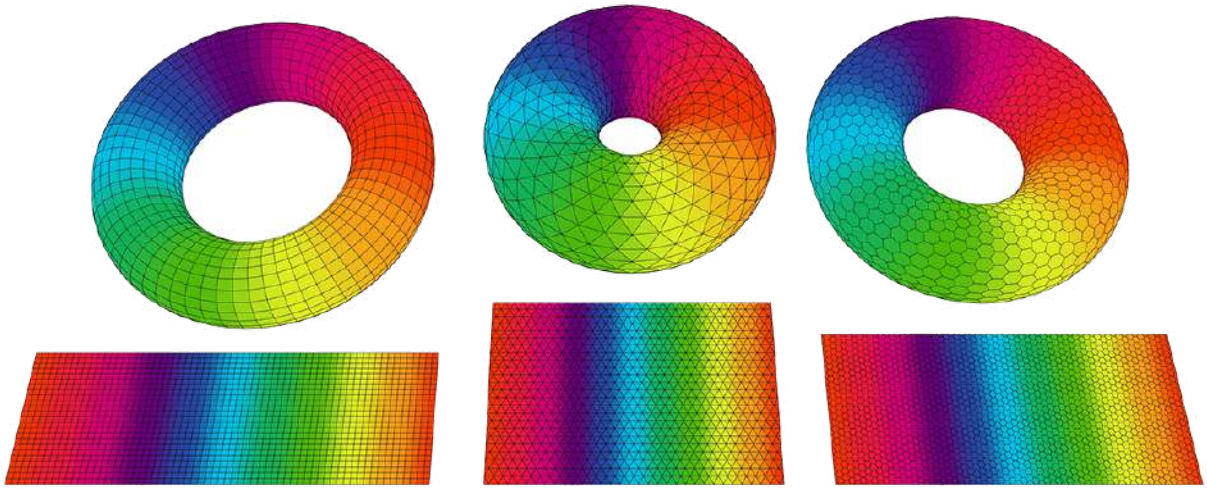
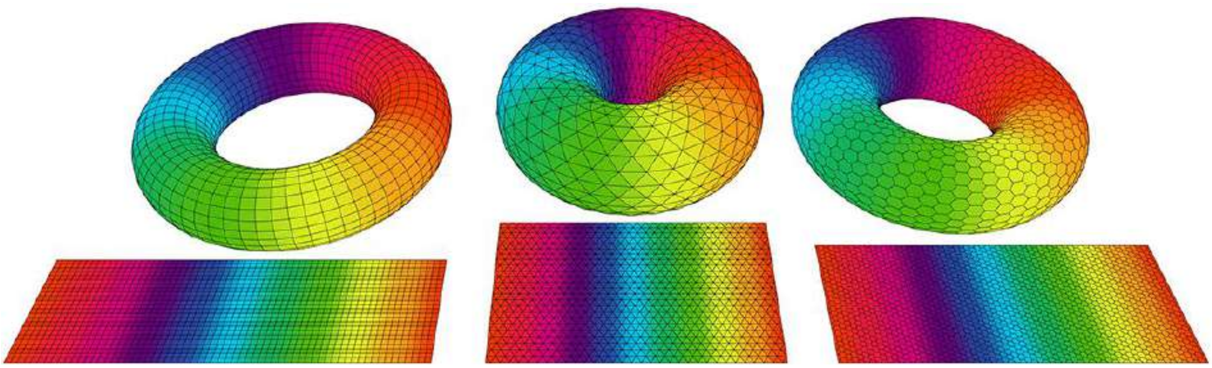


Figure 25: From Left to Right; Planar Quad, Tris and Hex Remeshings are transformed on tori surfaces through UV Mapping method.



3.3. Comparison 1

The 2D-Domains having the same surface area and the same total of panels differ in regards to their aspect ratio. (Quad 2D-Domain, 1:3; Tris 2D-Domain, 3:5; Hex 2D-Domain, 2:5. The aspect ratio given hereby are slightly rounded, see Fig.21)

In order to take aside the aforesaid difference, first, the 2D-Domains are mapped to generate 3D tori surfaces respecting their own aspect ratio. Needless to say, the observations are to be made out of such comparison will be free from the matters of fabricating one single surface. Basically, the purpose of this comparison is to analyse the intact behavior of each remeshing when they are limited with same surface curvature. It is also possible to consider the idea as having same amount of planar quad, tris and hex tiles made of rubber sheet like material and assembling them as a torus to observe the physical changes occurring on each tile (area deviation, edge length deviation… etc.) The resulting tori surfaces represented as mesh geometry is shown in Figure 25.

After mapping the tori out of all 2D-Domains, the metrics are evaluated respectively. The first metric is the area deviation. (see Fig.26) Besides the colouring of mesh faces, at this point, the statistical analysis is also included to have a better understanding of the deviations. (see Fig.27) In the charts, it is clear that the average and the median area values are precisely equal to one another for all the remeshings. Also notice that, even though the aspect ratio of the tori are different from one another, all of the resulting tori surfaces have precisely equal surface area in total.

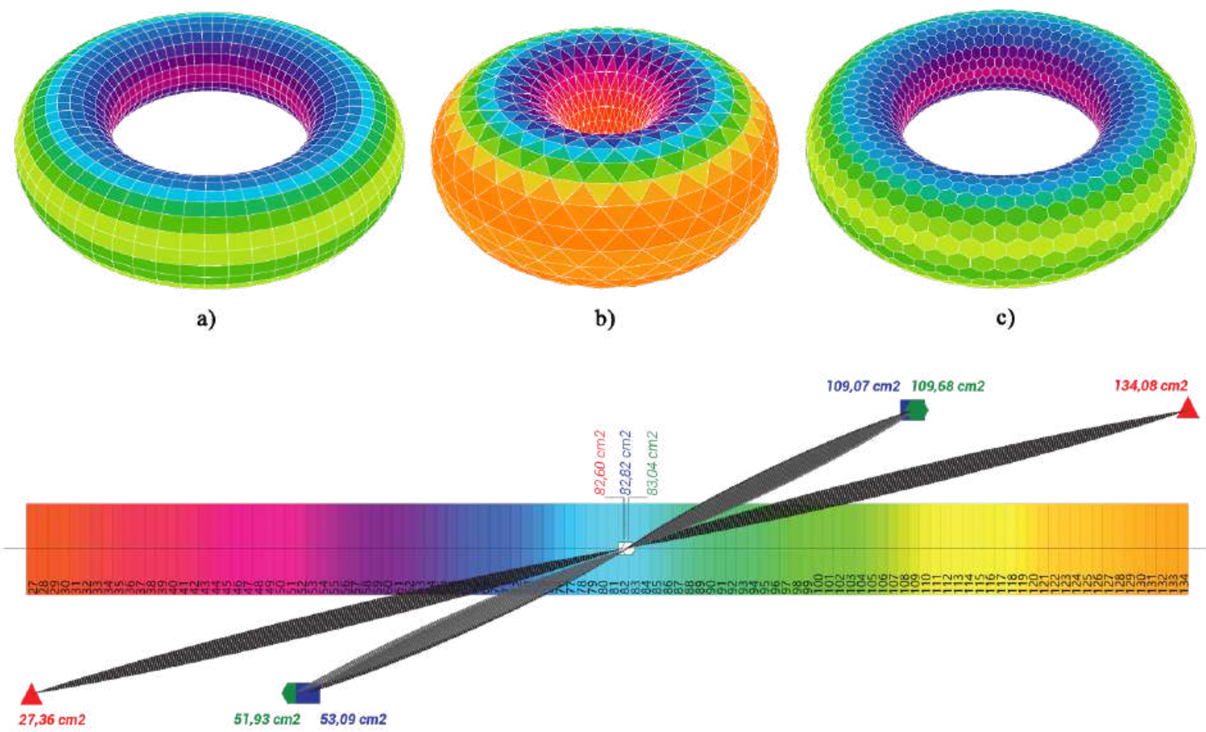


Figure 26: Area distribution of the remeshings represented with the legend for all ranges a) Quad b) Tris c) Hex

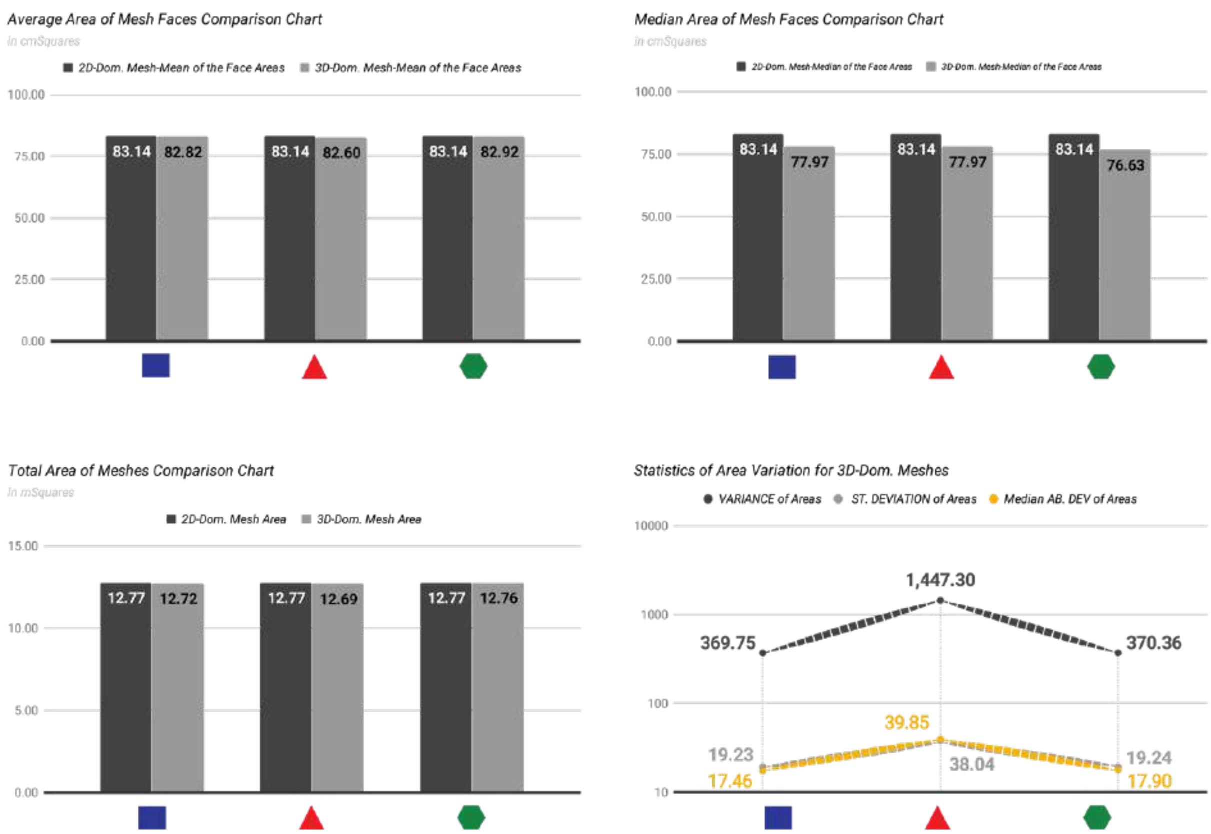


Figure 27: The statistics charts of the area deviation for Comparison 1

The second metric is the surface deviation. Besides the total distance comparison, also the standard deviation chart is included for a clear comparison. (see Fig.28-29)

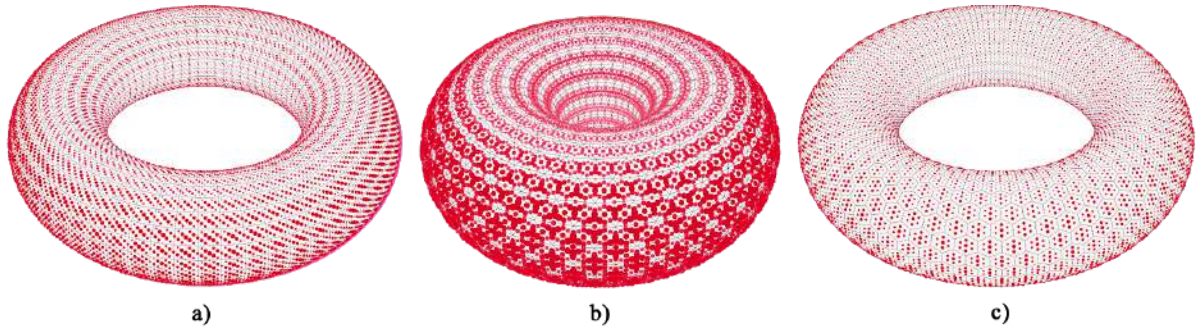


Figure 28: Surface fairness evaluation of Comparison 1 a) Quad b) Tris c) Hex

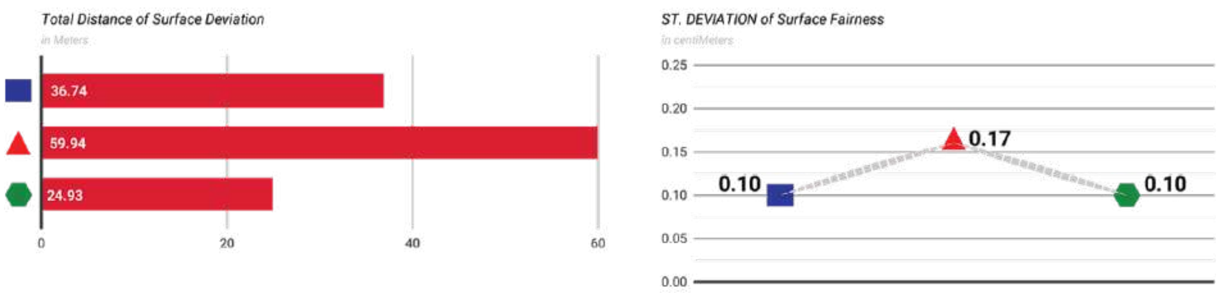


Figure 29: The statistics charts of the surface fairness for Comparison 1

Next, the planarity of each mesh is evaluated. Hereby, it is important to remind that no optimization is applied on either of the meshes and the comparison is limited by their inherent planarity properties. (see Fig.30)

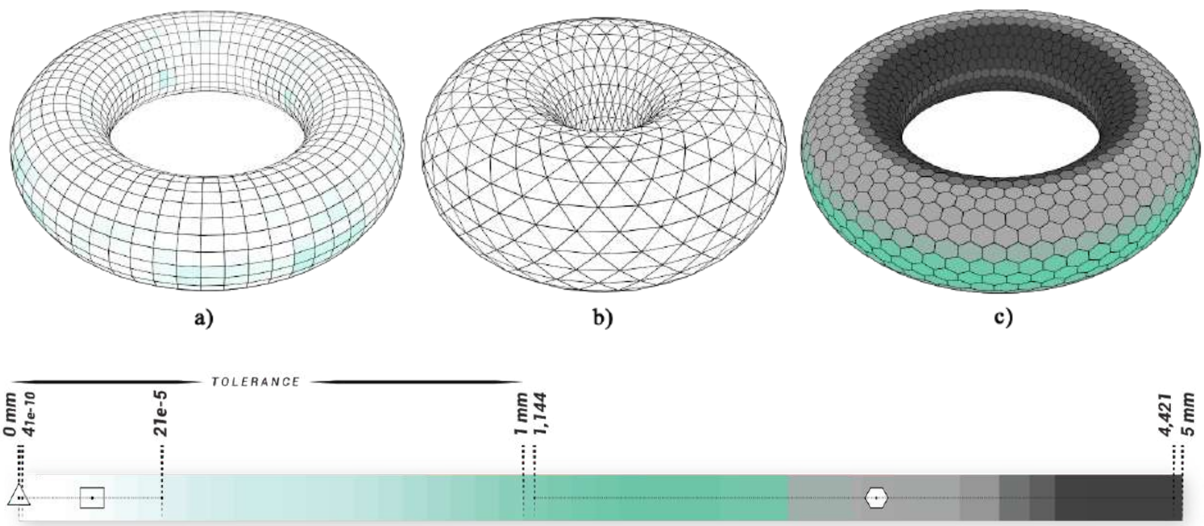


Figure 30: Planarity evaluation of Comparison 1 a) Quad b) Tris c) Hex

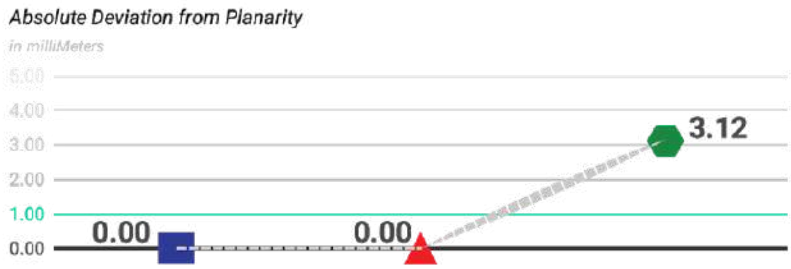


Figure 31: The average absolute deviation chart of planarity for Comparison 1

Lastly, the edge length and aspect ratio is evaluated and compared for all of the remeshings. Edge length histogram chart and statistics of aspect ratio is also presented. (see Fig. 32) For a better visual perception, the edges are first grouped in the bounds of the floor and ceiling integer values of their length, and then all given one colour denoting the domain they belong. For instance, such edge lengths as 3.45, 3.67, 3.21 cm ...etc. are grouped under the domain starting from 3cm. and ending at 4 cm. and they are all coloured as red both in the images.

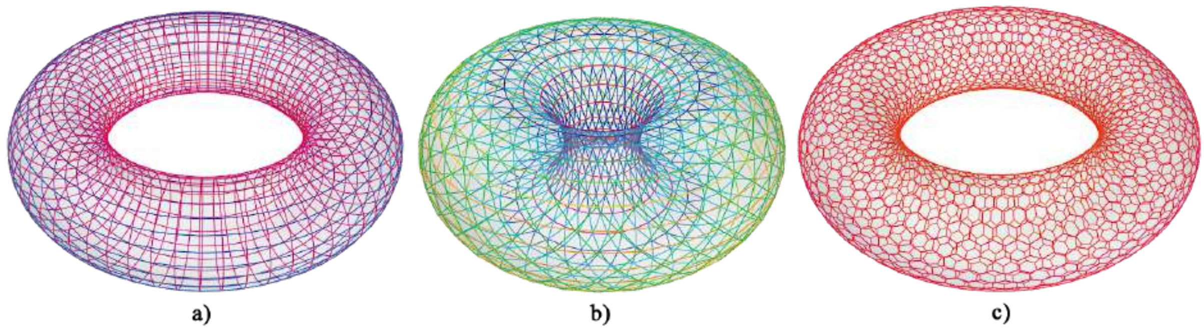


Figure 32: Edge Length evaluation of Comparison 1 a) Quad b) Tris c) Hex

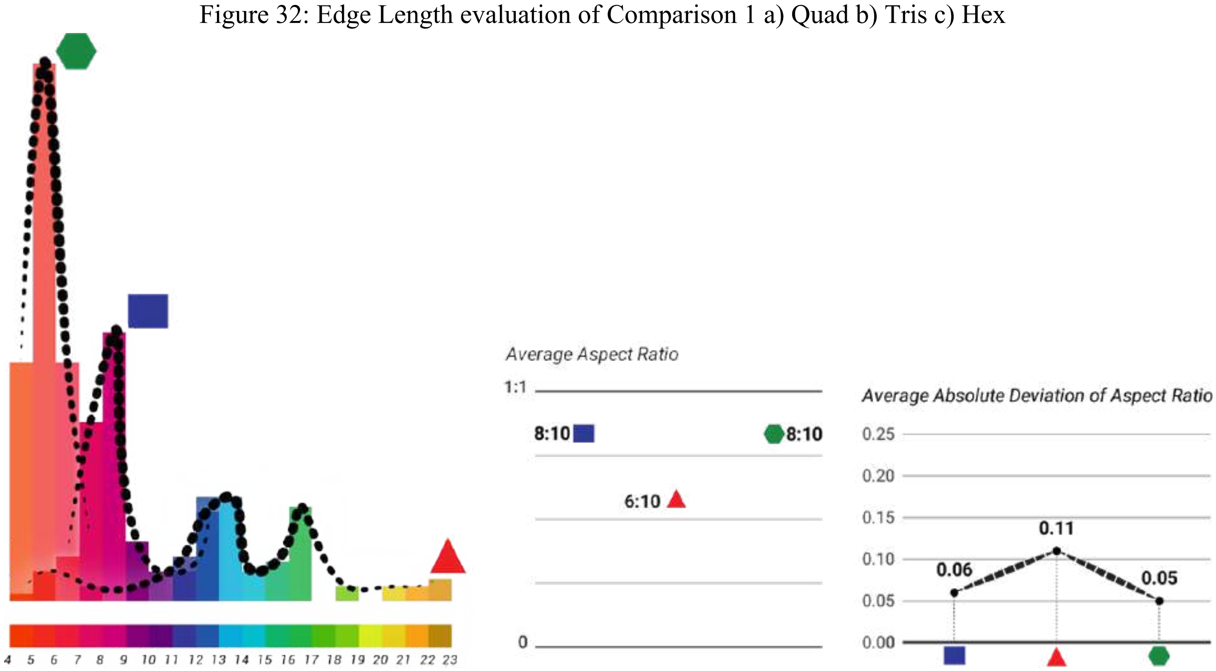


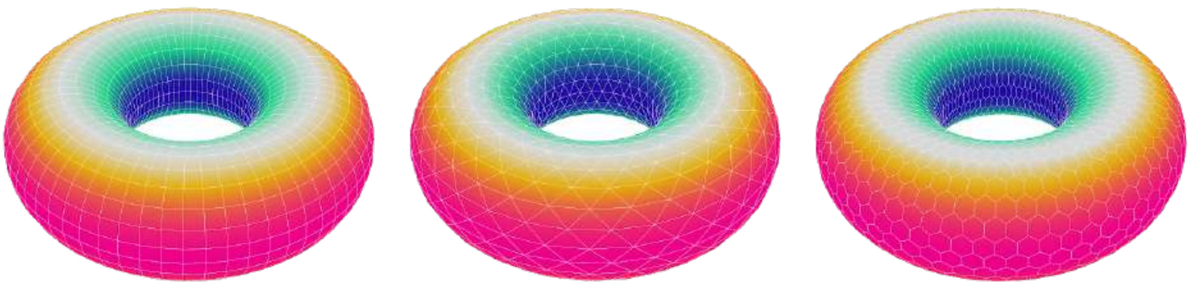
Figure 33: The statistics charts of edge length variation for Comparison 1.

3.3.1. Observations over Comparison 1

In terms of area deviation, the statistics show that Tris remeshing disperses two times more than Quad and Hex remeshings. It is observed that Quad and Hex have very similar deviation results. (see Fig.27)
In terms of surface deviation, Tris remeshing is the least fair remeshing to the surface while Quad and Hex remeshing follow it with exactly the same standard deviation value. (see Fig.29)
In terms of planarity, Tris is totally planar. Owing to the mesh generation method, and also to the surface characteristics, Quad remeshing negligibly deviates from planarity while Hex remeshing definitely requires some optimization to fit in the tolerable planarity range. (see Fig.31)
In terms of edge lengths, Hex remeshing has the minimum range varying from 4.25 to 7cm. Quad remeshing follows it by having several edge lengths between 6.61 and 12.97cm. Tris remeshing disperses the most pertaining to edge length range differing from 4.84cm to 22.82cm. The statistic charts also reinforces the aforesaid observation that Tris remeshing has the most scattered distribution of lengths. Quad follows it as the second remeshing with almost halved deviation value and Hex remeshing results in the most homogeneous distribution in terms of edge lengths. (see Fig.33)
In terms of aspect ratio variation, Tris remeshing has the maximum distortion. Quad and Hex remeshing are again quite similar.
As mentioned in the beginning, all of the remeshings have the same total of panels. To this extent, Tris remeshing has the minimum amount of nodes (768 in number) yet the vertex degree of each node is 6. Quad remeshing follows it with 1536 nodes in total all having valence of 4. Hex remeshing has the minimum vertex degree i.e. 3, however, the total of the nodes are 3072.
Overall, the results of Comparison 1 demonstrates that Tris can correlate to the figures of Quad and Hex remeshings when its total of panels are doubled. As mentioned before, the results in this section are observed over three different tori surfaces respecting the differences in the aspect ratio of 2D-Domains. From an architectural point of view, the subject of fabrication is always one certain surface. Therefore, the second comparison evaluates the results of the 2D-Domains mapped to one single torus.

3.4. Comparison 2

The second comparison is run by mapping the 2D-Domains on one single torus which is the surface T used in metric definitions section.(Section 1.2) All of the metrics are evaluated in the same order as the comparison above.



a) b) c)

Figure 34: Meshes mapped on the surface T a) Quad b) Tris c) Hex. All of the meshes are coloured due to Kg.

Before proceeding to the observations, it is important to note here that the figures of area deviation are consistent with the contention of the study whereas all of the deviations are precisely equal to one another. This result can also be easily predicted from the colouring of mesh faces done in respect to the area distribution. (see Fig. 35-36)

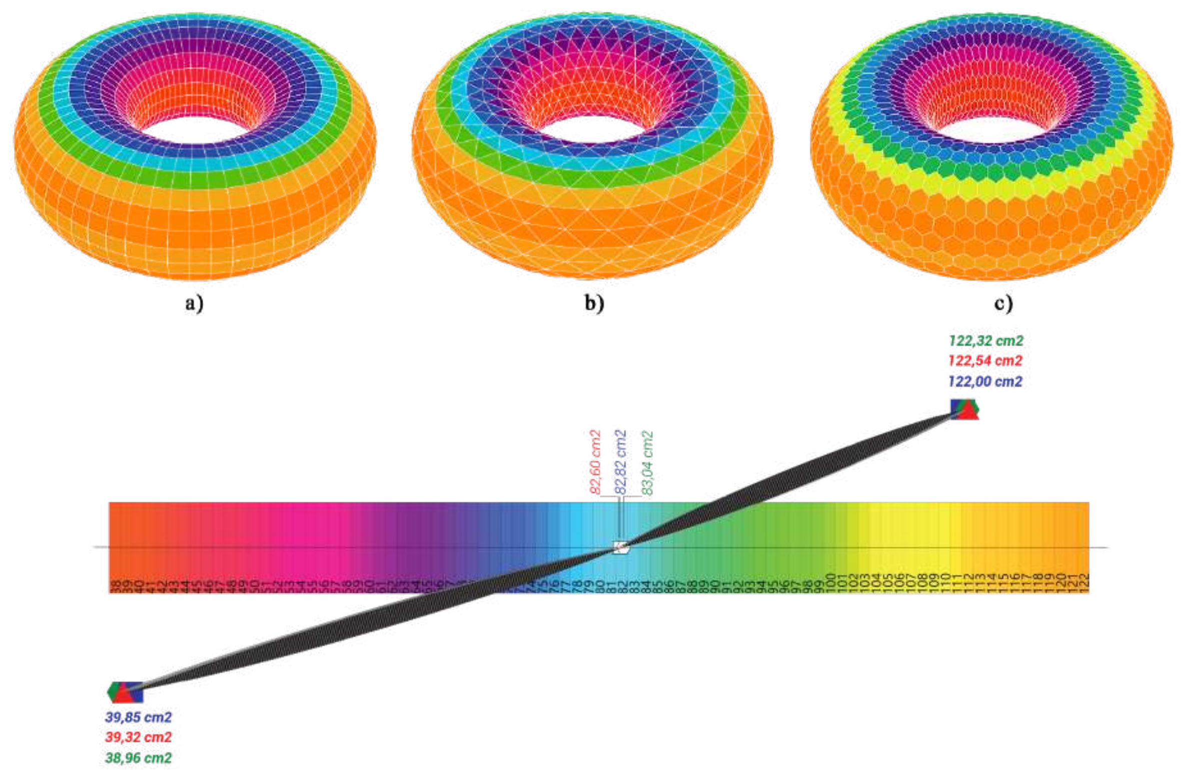


Figure 35: Area distribution of the remeshings represented with the legend for all ranges a) Quad b) Tris c) Hex

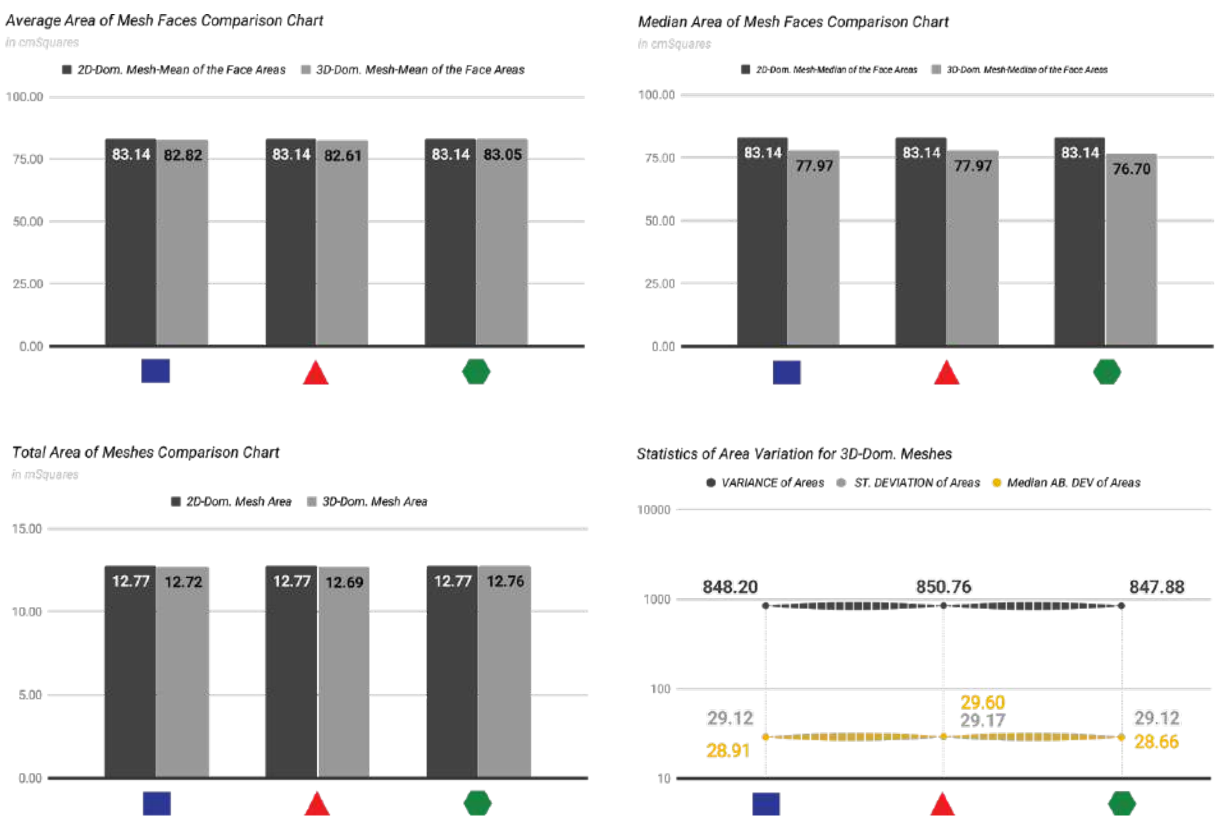


Figure 36: The statistics charts of the area deviation for Comparison 2.

3.4.1. Observations over Comparison 2

The remeshings mapped on the surface *T* have precisely equal average and median value of the areas. The statistical measurements for area deviation is almost equal to one another.

In terms of surface deviation, Tris remeshing is the least fair remeshing to the surface while Quad and Hex remeshing follow it with exactly the same standard deviation value. More importantly, comparing to the first observations, the surface deviation number for Tris has slightly fallen while Quad and Hex figures have risen in correlation to the findings of Comparison 1. (see Fig.37-38)

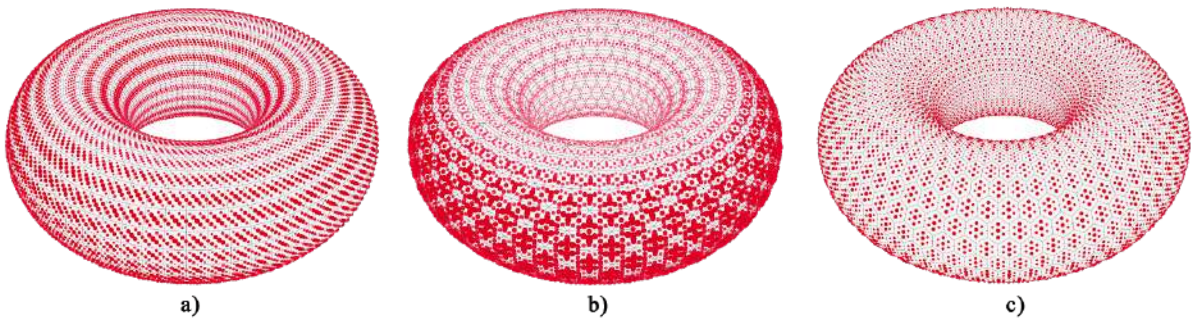


Figure 37: Surface fairness evaluation of Comparison 2 a) Quad b) Tris c) Hex

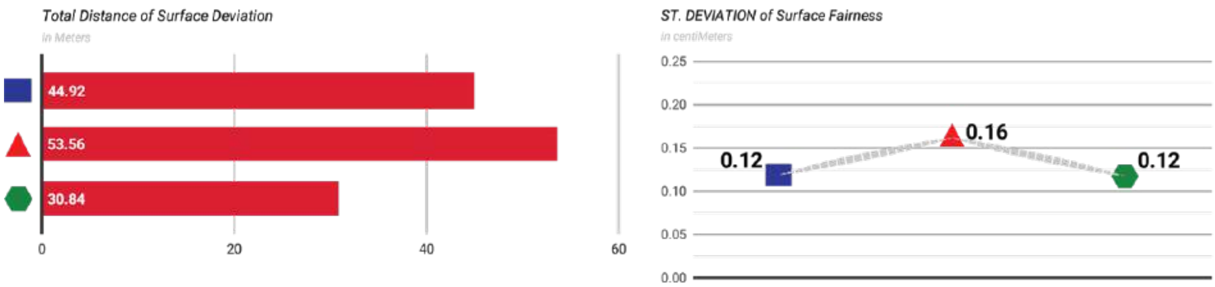
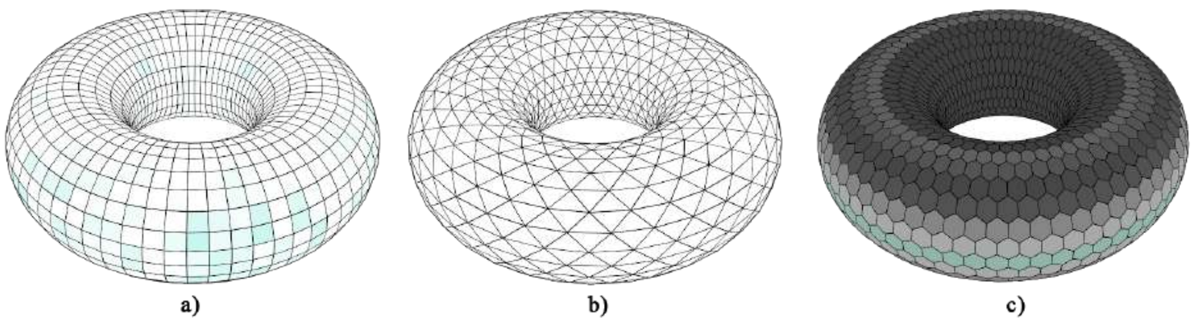


Figure 38: Statistics for the surface fairness of Comparison 2

In terms of planarity, the general results are all consistent with Comparison 1. On surface *T*, Quad remeshing reaches even better planarity while the deviation of Hex remeshing increases. It is important to remind again that all of the meshes are generated with exact same implementation. (see Fig.39-40)



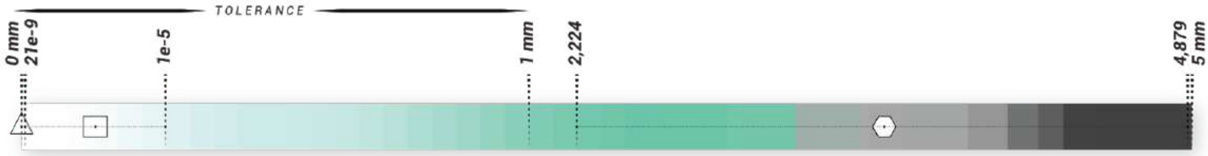


Figure 39: Planarity evaluation of Comparison 1 a) Quad b) Tris c) Hex

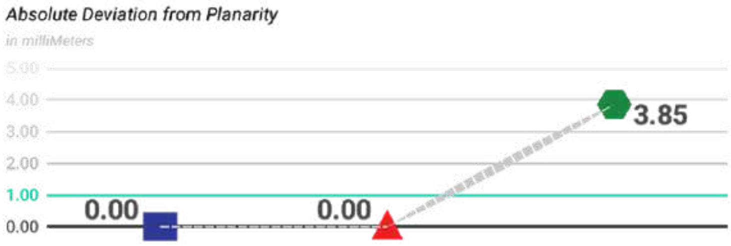
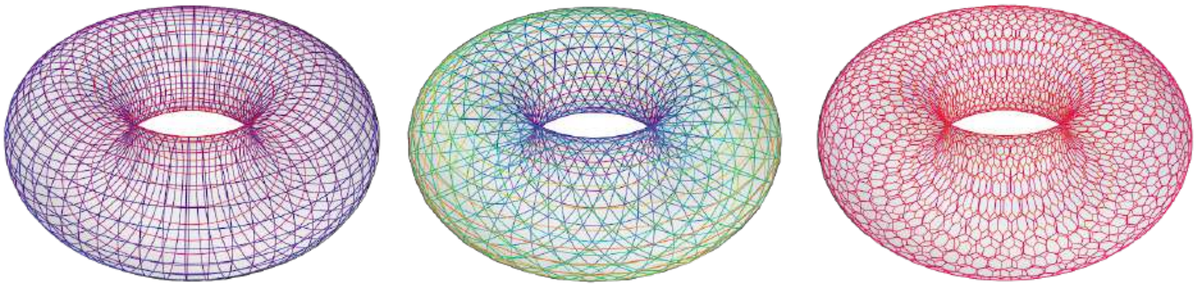


Figure 40: The average absolute deviation chart of planarity for Comparison 2

In terms of edge lengths, Hex remeshing has the minimum range varying from 3.85 to 7.36cm. Quad remeshing follows it by having several edge lengths between 3.98 and 11.85cm. Tris remeshing disperses the most pertaining to edge length range differing from 7.95cm to 23.67cm. The standard deviation values also point to the same conclusion where Tris remeshing has the most scattered distribution of lengths. Quad follows it as the second remeshing and Hex remeshing results in the most homogeneous dispersion in terms of this metric. (see Fig. 41-42)



a) b) c)

Figure 41: Edge Length evaluation of Comparison 2 a) Quad b) Tris c) Hex

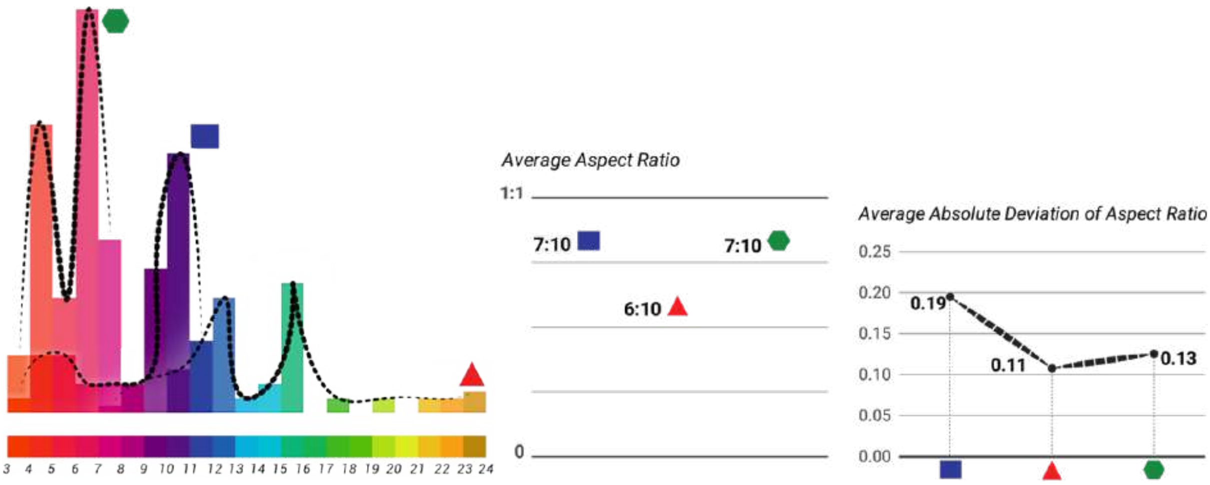


Figure 42: The statistics charts of edge length variation for Comparison 2.

In terms of aspect ratio variation, the average absolute deviation statistics show that the number has stayed the same for Tris remeshing while the average absolute distortion has doubled for Hex remeshing. The noteworthy results is that Quad panels are distorted the most where the average absolute figure is seen to triple when compared to Comparison 1 results. (see Fig.42)

To sum up, when mapped on the same torus surface, Tris remeshing, again, results in comparingly less attracting figures while Quad remeshing come to the fore as the optimum choice in regards to overall metric findings. It is also crucial to underline that Hex remeshing adapts the surface curvature comparingly better than the rest. At this juncture, it is still not very clear whether the results obtained from the previous two comparisons may vary under the terms where the aspect ratio of the 2D-Domains were also totally equal or not. This uncertainty remains as a matter of further research. Also, up until this section, none of the remeshings compared are optimized. Therefore, the next section will compare the meshes representing the surface *T* with their planarized versions.

4. Planarization

In this section, first the planarization step for each remeshing is explained. Then, as aforesaid, the results are compared between pre-optimization and post-optimization meshes. Further down, optimized meshes is mentioned as PQuad, PTris and PHex in short. For planarization, Kangaroo Physics[6] is used.

4.1. Implementation

The main goal for the optimization of Quad and Hex remeshings is to make the vertices of each face planar. For Quad remeshing, the planarization is checked up on the planarity statistic at each iteration as some of the converged results were observed to be regressing when compared to the planarity numbers of the input mesh. This occurs as a consequence of the input mesh being almost planar and the strength of the planarity goal is supposed to be increased drastically until the resulting mesh measures better planarity. Additionally, the goal for equalizing the panel areas is introduced to decrease the relevant deviation for the Quad mesh itself.

For Hex remeshing, only the planarity goal is introduced. Tiling a surface with the same kind of polygons is comparingly more challenging to the same kind of tiling on planar surfaces. The variation of face shapes is inevitable since they are constrained by surface curvature.[7] Therefore, as a result of planarization process, significant shape variations of the Hex faces are expected in the case of representing the surface *T*. More precisely, when all the vertices of hexagonal mesh faces reach the state of being planar, regular hexagons appear on synclastic curvature while the faces corresponding to anticlastic curvature appear as bow-tie like mesh faces. On planar regions, hexagons take the shape of quadrilaterals. As a nuance, it may be important to mention the exceptional results observed during the planarization of Hex remeshing depending on the orientation of the tiling on the surface *T*. (see Fig.43)

For Tris remeshing, the main purpose is the equalization of edge lengths and the goal is applied as forcing the incircles of triangular faces to become tangent to one another in the optimization . Also, in this case, the area goal is also used to unify the panel areas. In the following section, the comparison 3 is discussed between the input and output meshes of the planarization process.

4.2. Comparison 3

In this section, first of all, the planarity metric is evaluated. (see Fig.44-45) In regards to the flatness tolerance assumed, Quad and Hex remeshings are successfully optimized as planar meshes. The rest of the metrics are then evaluated on PQuad, PTris and PHex meshes.In the representation of the metrics, the upper row represents the input meshes while the lower row shows the optimized meshes.

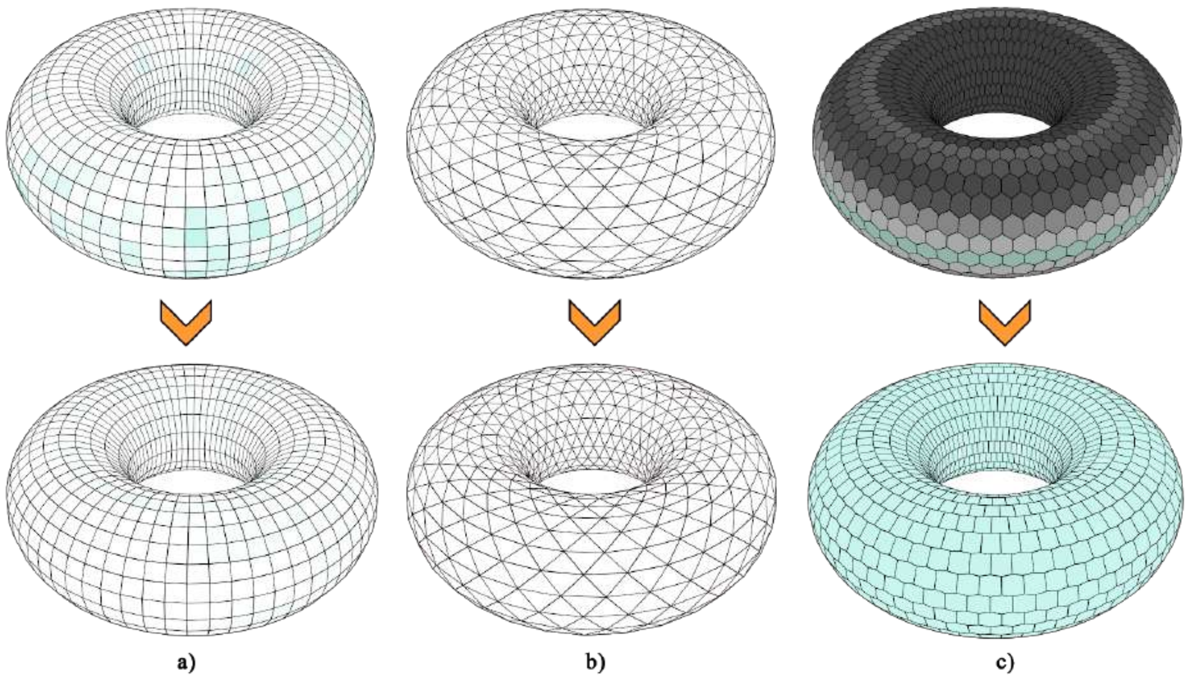


Figure 44: Planarity evaluation of Comparison 3.
a) Quad- PQuad b) Tris- PTris c) Hex- PHex

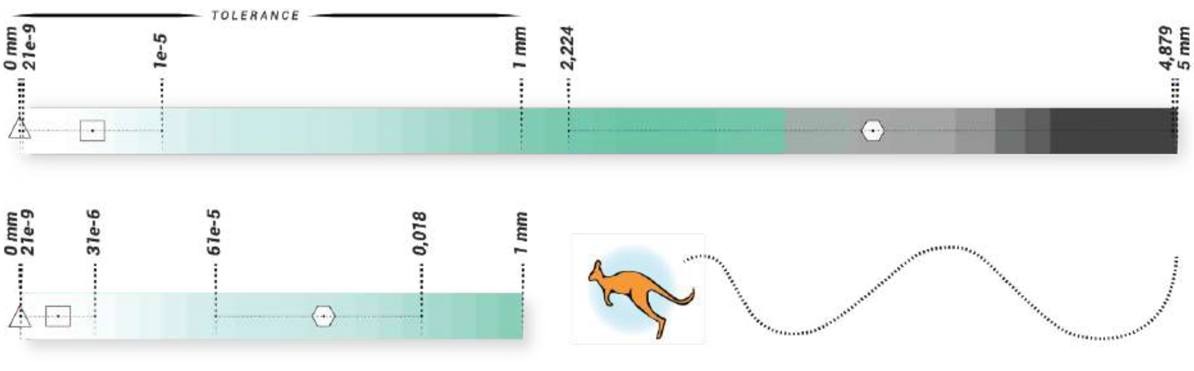


Figure 45: The comparison between the planarity range of input and output meshes.

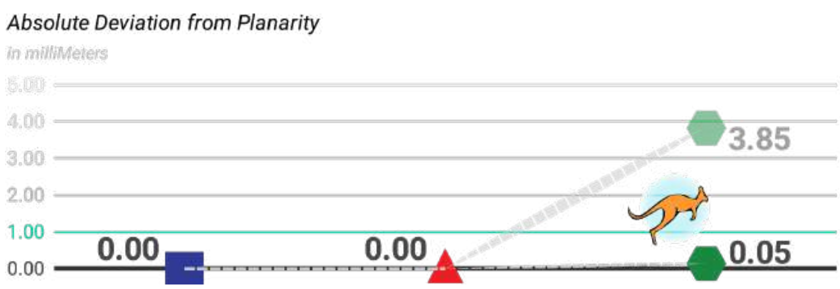


Figure 46: The average absolute deviation chart of planarity for Comparison 3.

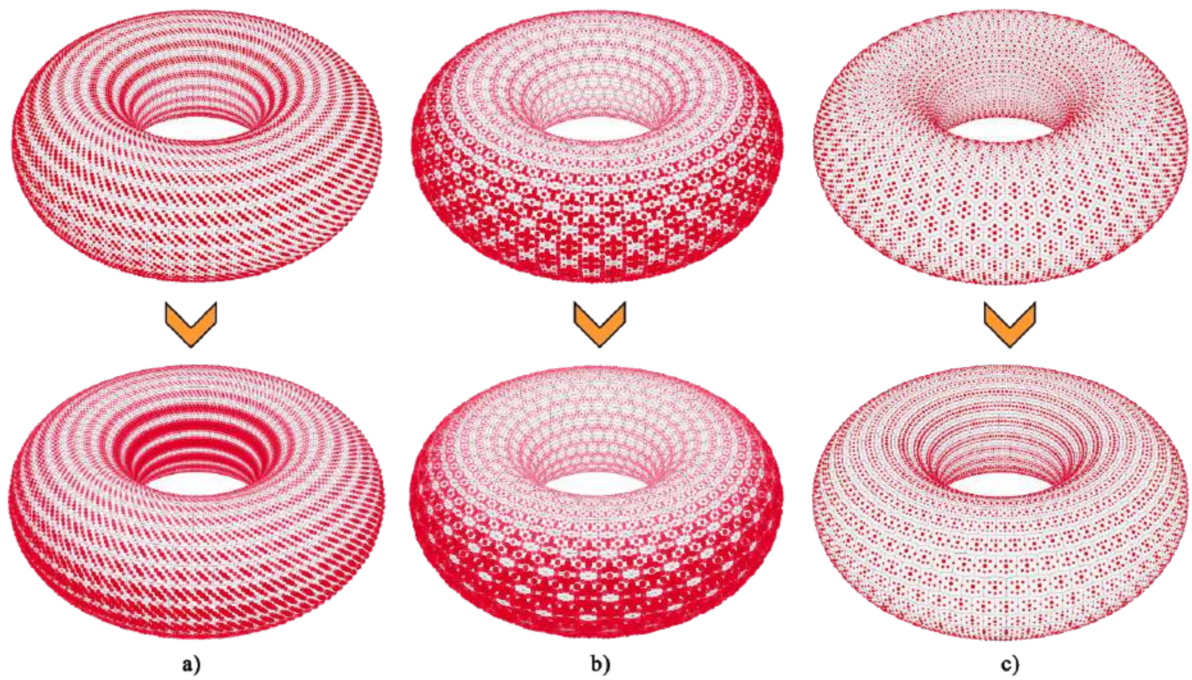


Figure 47: Surface fairness evaluation of Comparison 3.
a) Quad- PQuad b) Tris- PTris c) Hex- PHex

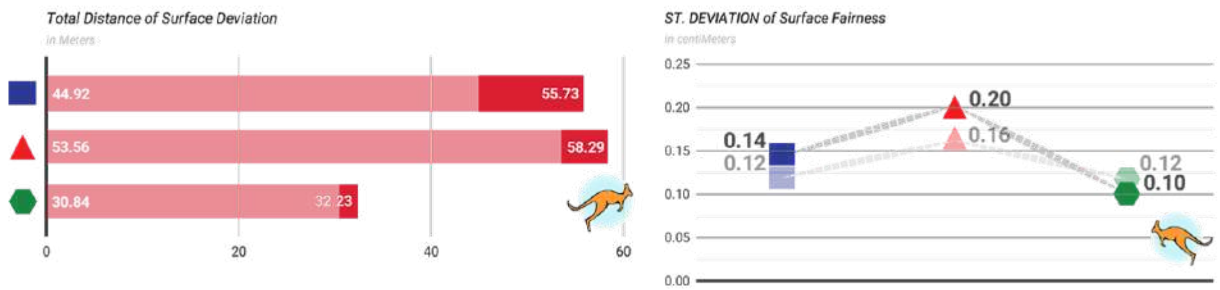


Figure 48: Statistics for the surface fairness of Comparison 3

Evidently, the surface deviation is increased for both Quad and Hex remeshings as they are forced to become more planar representations of the surface *T*. The surface deviation for Tris remeshing is increased as the edge lengths are forced to be unified. Having a look at the fairness representation of PTris, (see Fig.47) it is seen that the surface deviation occurring at the anticlastic curvature is decreased while the faces around the equator line is stretched through the poloidal axis and hence, deviate from the surface *T* comparingly more than its non-optimized counterpart. Apparently, the increase on the synclastic part is larger than the decrease on the anticlastic region, thereby the total deviation goes up. This change occurs as the mesh faces move from the equator line towards the inner circle in regards to the optimization of edge lengths. Here also notice that the area goal introduced to the optimization of Tris causes the same movement of the faces in the opposing direction. To be more clear, the movement happens from the inner circle towards the equator line in order for the mesh faces to have similar areas.

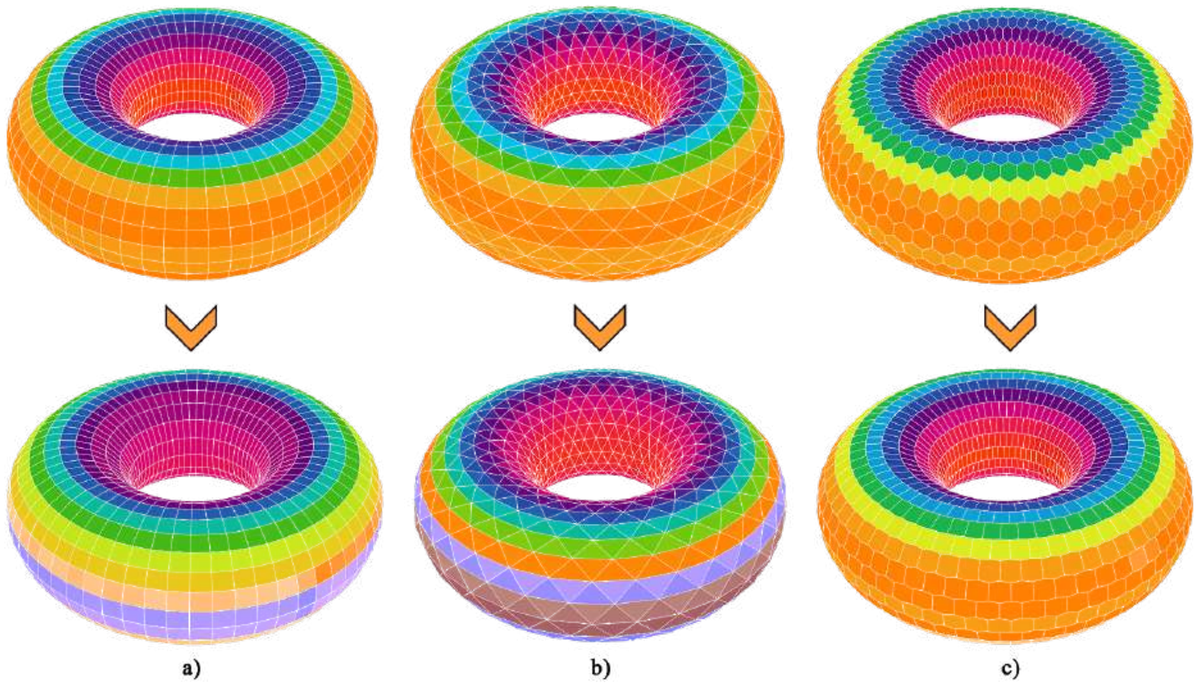


Figure 49: Area distribution evaluation for Comparison 3.
a) Quad- PQuad b) Tris- PTris c) Hex- PHex.

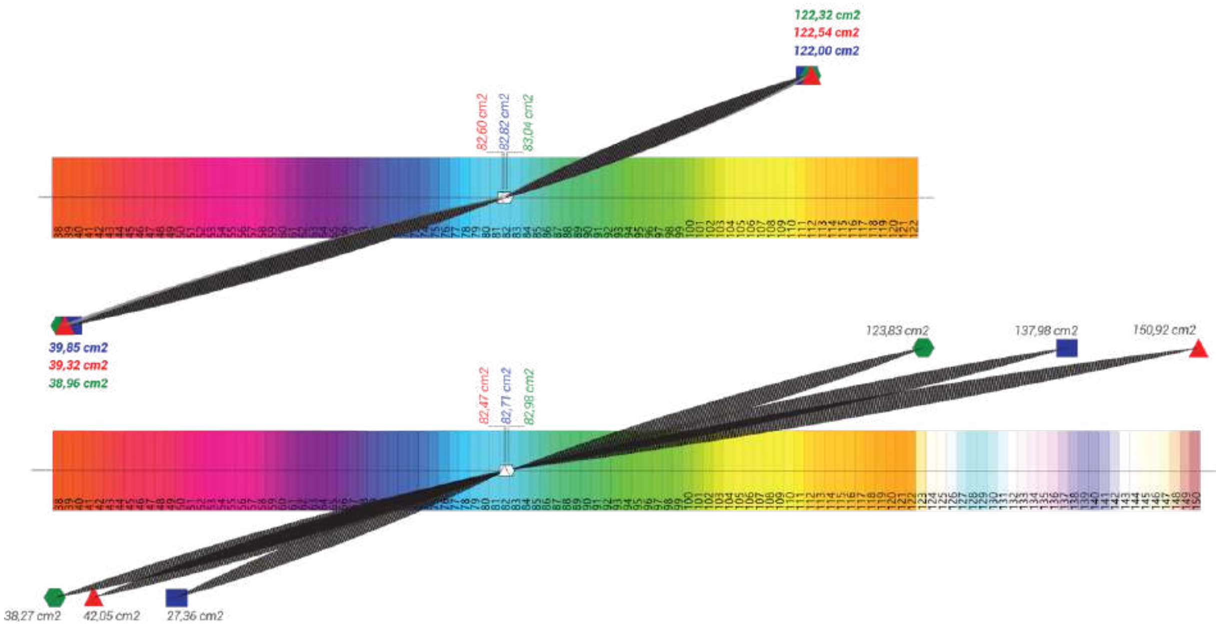


Figure 50: The comparison between the area range of input and output meshes.

According to the statistics, (see Fig.51) it is clearly seen that the standard deviation of the area values is increased for PQuad and PTris meshes. The goal which is used to unify the panel areas only helps to reduce the median absolute deviation as a consequence of the surface characteristics. As aforesaid, torus is a rotational surface. Thereby, the movement for unifying the areas only occurs in poloidal direction and is restricted on toroidal direction. This fact also causes much distortion of the panels as transmuting the mesh into comparingly longer and thinner faces. Since no area goal is used for optimizing the Hex remeshing, no significant change occurs in terms of area deviation. To this respect, it can be stated that in this particular case, the goal for unifying the areas is not necessary to include in the rationalisation process. However, the noteworthy aspect of the goal is that, opposing to the standard deviation figures, the numbers of the median absolute deviation has fallen for Quad and Tris remeshings, (deviation for Quads has almost halved) meaning the area values are more clustered together after the optimization. It is important to note here that the standard deviation is more affected by outliers (extremely low and extremely high values) of the data set and hence may point to less robust results in case non-normality.

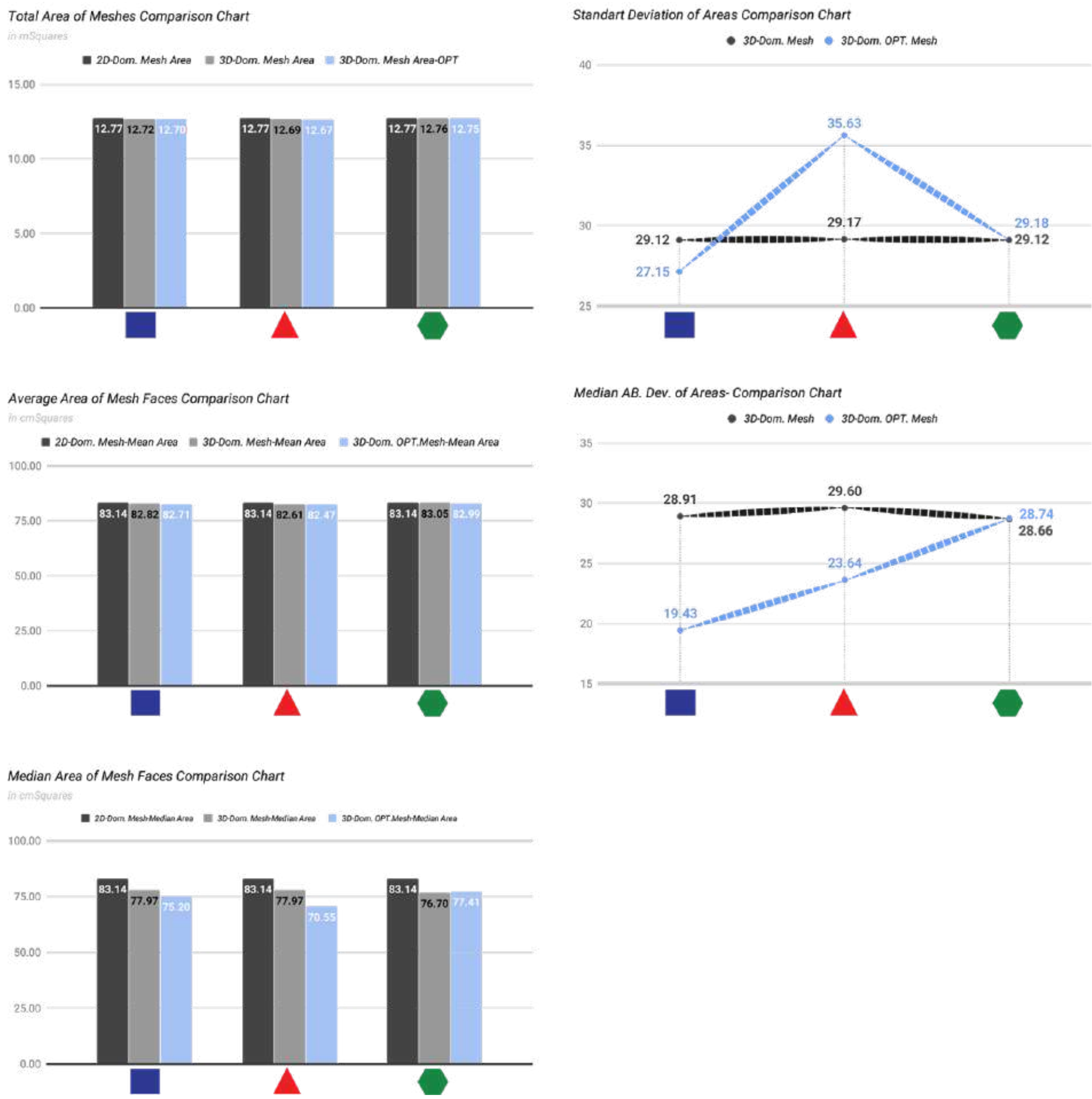


Figure 51: The statistics charts of the area deviation for Comparison 3.

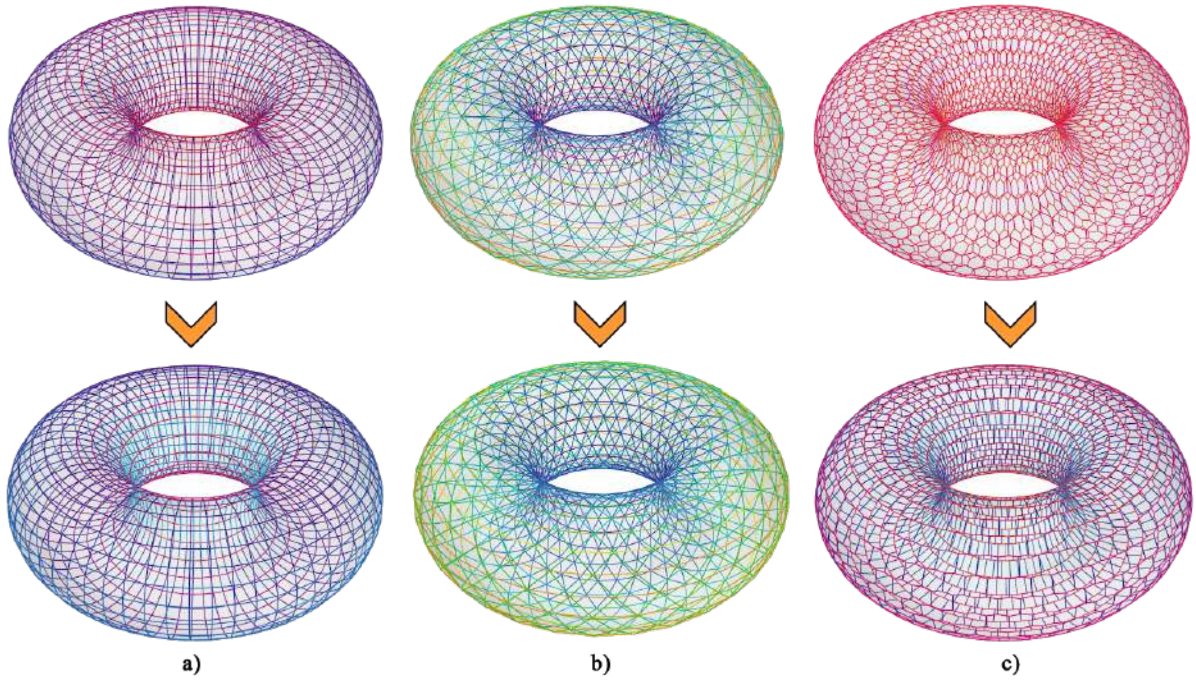


Figure 52: Edge Length evaluation of Comparison 3
a) Quad- PQuad b) Tris- PTris c) Hex- PHex.

In terms of edge lengths, Hex remeshing has the minimum range varying from 1.92 to 11.28cm. Quad remeshing follows it by having several edge lengths between 3.92 and 12.95cm. Tris remeshing disperses the most pertaining to edge length range differing from 7.92cm to 23.67cm. The standard deviation values also point to the same conclusion where Tris remeshing has the most scattered distribution of lengths. Also compared to their non-optimized counterparts, only Tris remeshing deviates more in terms of edge lengths variation. Quad follows it as the second remeshing and Hex remeshing results in the most homogeneous dispersion in terms of this metric. (see Fig. 52-53-54)

In terms of aspect ratio, the results are quite the opposite compared to edge length variations. As expected, a significant distortion is observed at PHex panels as far as the planarization of hexagons causes a drastic change in panel shapes. However, statistically speaking, PQuad panels is distorted even more in consequence of the area goal used in the optimization. As for Tris, even though the edge length variance results is the maximum spread, the panel distortion is the minimum figure amongst all.

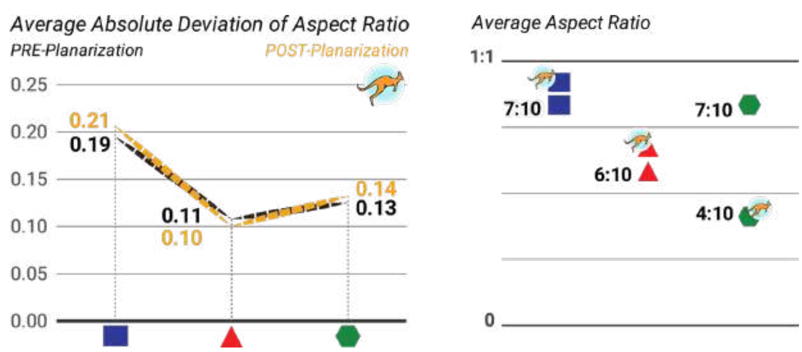


Figure 53: The statistics charts of edge length variation for Comparison 3

5. Conclusions and Future Research

The results of this study is bounded with Tori surface characteristic and may vary in the context of different surfaces. Taking into account the statistical data, in terms of rationalisation, the least interesting results are observed for Tris remeshing. Some exceptional findings, such as the minimum distortion of panels in comparison 3 or the fact that the geometry itself inherently being totally planar is comparingly minor positive aspects to the rest of the meshings. Also it has the most complex nodes as valence of 6.Apart from the examined features in this study, triangular meshes does not have a proper definition of offset meshes. [Pottman *et. al.* 2007b]
Hex remeshing has great capability of adapting to the surface curvature. The major issue about hexagonal remeshings is the planarization of the geometry.
Quad remeshing comes to the fore as the optimum representation of the torus surface, having superiority to the rest of the meshes in almost all of the metrics compared at each and every comparison. It is noteworthy that Quad and Hex remeshing has very alike figures in the case study.
To sum up, the methodology followed in this study should be examined on different surfaces as to evaluate its potential to equalize the area deviation for the three remeshings. Also in the study, it is still not clear whether the difference in the aspect ratio of the 2D-Domains have any effects on the statistics or not.
As mentioned above, in the further studies those problems will be researched. Also the necessity of the singularities in terms of better rationalisation results will be studied.

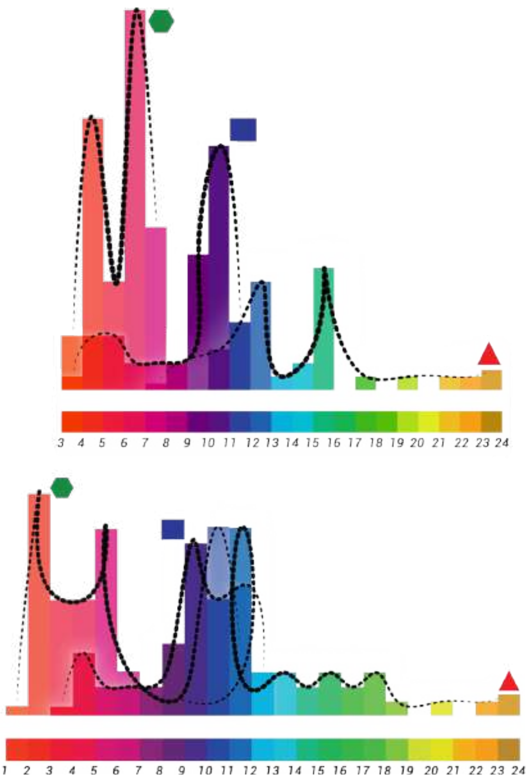


Figure 54: Histogram charts of edge lengths for Comparison

References

[1] J. Burry, M. Burry, *The New Mathematics of Architecture.* London:Thames & Hudson, 2012

[2] M. Fabré Nadal, *"Anisotropic Remeshing Methodologies: State of Art and Performing of Planar Polygonal Anisotropic Remeshing Applied to Architecture"*, Sep. 2018

[3] H. Pottman, A. Asperl, M. Hofer, D. Bentley, *Architectural Geometry*. Exton, PA.: Bentley Institute Press, 2007

[4] N. Dunn, *Digital Fabrication in Architecture.* London: Laurence King Publishing, 2012

[5] A. Nejur, S. Kyle, "Ivy: Progress in Developing Practical Applications for a Weighted-Mesh Representation for Use in Generative Architectural Design", in *ACADIA 2017: DISCIPLINES & DISRUPTION Proceedings of the 37th Annual Conference of the Association for Computer Aided Design in Architecture* Cambridge, MA 2-4 November, 2017), pp. 446- 455

[6] D. Piker, "Kangaroo: Form Finding with Computational Physics." *Architectural design,* vol. 83, Issue 2, pp. 136-137, Mar. 2013

[7] W. Wang, Y. Liu, D. Yan, B. Chan, R. Ling, F. Sun, *" Hexagonal Meshes with Planar Faces"*, Jan. 2008

[8]